Bayesian Linear Models

PUBH 8442: Bayes Decision Theory and Data Analysis

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Linear model

- For observations $y_1, \ldots, y_n$, the basic linear model is

$$y_i = x_{1i}\beta_1 + ... + x_{pi}\beta_p + \epsilon_i,$$

  - $x_{1i}, \ldots, x_{pi}$ are predictors for the i^{th} observation.
  - ϵ_i are error terms.

- In matrix form:

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

  - $\mathbf{y} = (y_1, \ldots, y_n)$, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$
  - X is the matrix with entries $X_{ij} = x_{ij}$

Linear model

- Assume X is fixed (non-random)
- Assume errors are normal and iid with equal variance:

$$\boldsymbol{\epsilon} \sim \text{Normal}(\mathbf{0}, \sigma^2 I).$$

- Standard frequentist estimates are

$$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y} \text{ and } \sum_{i=1}^{n} (y_i - X_i \hat{\beta})^2$$

$$\hat{\sigma}^2 = s^2 = \frac{1}{n-p} (\mathbf{y} - X\hat{\beta})^T (\mathbf{y} - X\hat{\beta}).$$

  - These estimates are unbiased, and can be motivated by least-squares.

- Under a Bayesian framework, we put a prior on β and σ^2.

Uninformative priors

$$\vec{x} \sim N(\vec{\mu}, \Sigma), \quad p(\vec{x}) \propto \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right)$$

- Consider uniform prior for β and Jeffreys prior for σ^2:

$$p(\beta, \sigma^2) \propto \frac{1}{\sigma^2}.$$

- The posterior for β, given σ^2, is

$$p(\beta \mid \mathbf{y}, \sigma^2) = \text{Normal}\left(\hat{\beta}, \sigma^2 (X^T X)^{-1}\right)$$

$$p(\vec{\beta} \mid \vec{y}, \sigma^2) \propto p(\vec{y} \mid \beta, \sigma^2) \cdot \underline{p(\beta, \sigma^2)} \qquad p(\beta,\sigma^2) \propto \frac{1}{\sigma^2} \propto 1$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}(\vec{y}-X\beta)^T(\vec{y}-X\beta)\right\}$$

$$\therefore \propto \exp\left\{-\frac{1}{2}\left(\beta-(X^TX)^{-1}X^Ty\right)^T \left(\sigma^2 (X^TX)^{-1}\right)^{-1} \left(\beta-(X^TX)^{-1}X^T\vec{y}\right)\right\}$$

Uninformative priors

- The marginal posterior of σ^2 is

$$p(\sigma^2 \mid \mathbf{y}) = IG\left(\frac{n-p}{2}, \frac{(n-p)s^2}{2}\right)$$

- Equivalently:

$$\sigma^2 \sim \frac{(n-p)s^2}{U} \quad \text{where } U \sim \chi^2_{(n-p)}.$$

Uninformative priors

- The marginal posterior for β_i is a non-central t-distribution:

$$\frac{\beta_i - \hat{\beta}_i}{s\sqrt{(X^T X)^{-1}_{ii}}} \sim t_{n-p}.$$

- For a new predictor vector $\mathbf{x}_{(n+1)}$, the posterior predictive for y_{n+1} is also a non-central t-distribution:

$$\frac{y_{n+1} - \mathbf{x}_{n+1}\hat{\beta}}{s\sqrt{1 + \mathbf{x}_{n+1}(X^T X)^{-1}\mathbf{x}_{n+1}}} \sim t_{n-p}.$$

- All given results for $p(\beta, \sigma^2) \propto \frac{1}{\sigma^2}$ correspond to standard frequentist inference for linear regression!

Example: Body Fat

- The % body fat ($BF\%$) is measured for 100 adult males. [1]
  - Using sophisticated and precise technique (water immersion)
- Also measure the following for each person:
  - 1: *Age* (in years)
  - 2: *Weight* (in pounds)
  - 3: *Height* (in inches)
  - Circumference of the *neck* (4), *chest* (5), *abdomen* (6), *ankle* (7), *bicep* (8), and *wrist* (9) in cm.
- Data available at `http://www.lock5stat.com/datasets1e/BodyFat.csv`
- Would like to predict $BF\%$ from the 9 additional measurements

[1] Johnson, R. "Fitting Percentage Body Fat to Simple Body Measurements," *Journal of Statistics Education*, 1996.

Example: Body Fat

- Assume $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_{100})$ give $BF\%$ for subjects $1, \ldots, 100$
  - $\bar{\tilde{y}} = 18.6\%$
  - $s_{\tilde{y}} = 8.01\%$
- Let $X : 100 \times 9$ be the matrix of standardized predictors

$$X_{i,j} = \frac{\tilde{x}_{i,j} - \text{mean}(\tilde{\mathbf{x}}_{\cdot,j})}{\text{stdev}(\tilde{\mathbf{x}}_{\cdot,j})}$$

  - $\tilde{X}_{i,j}$ is measurement j (unstandardized) for subject i
- The mean BF% for american adult men is 18.5%
- For $\mathbf{y} = \tilde{\mathbf{y}} - \mathbf{18.5}$ consider the model

$$\mathbf{y} = \beta X + \epsilon$$

- Assume $\epsilon \sim \text{Normal}(\mathbf{0}, \sigma^2 I)$
- Use uninformative prior:

$$p(\beta, \sigma^2) = \frac{1}{\sigma^2}$$

- Recall $p(\beta_i \mid \mathbf{y})$ is a non-central t:

$$\frac{\beta_i - \hat{\beta}_i}{s\sqrt{(X^TX)^{-1}_{ii}}} \sim t_{91}.$$

$\beta_i = s\sqrt{(X^TX)^{-1}_{ii}}\, t_{91} + \hat{\beta}_i$

UB: use $t_{91, 0.975}$

LB: $t_{91, 0.025}$

where

$$\hat{\beta} = (X^TX)^{-1}X^T\mathbf{y}$$

and

$$s = \sqrt{\frac{1}{91}||\mathbf{y} - X\hat{\beta})||^2} = 4.11$$

Example: Body Fat

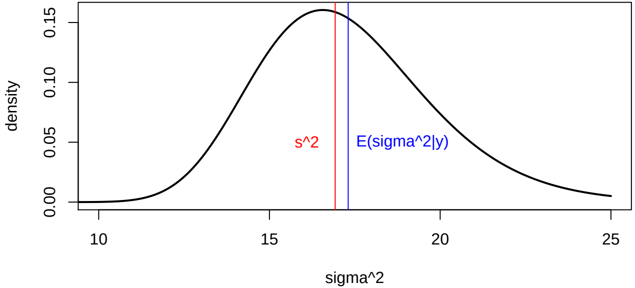
- Estimates and 95% credible intervals for $\beta_i' s$:

Variable	$\hat{\beta}_i$	95% credible interval
Age	0.956	(-0.186, 2.099)
Weight	-2.458	(-7.397, 2.480)
Height	0.097	(-1.328, 1.523)
Neck	0.002	(-1.727, 1.732)
Chest	-1.181	(-3.889, 1.526)
Abdomen	10.597	(7.639, 13.554)
Ankle	0.304	(-1.137, 1.745)
Biceps	0.454	(-0.935, 1.844)
Wrist	-2.201	(-3.807, -0.596)

http://www.ericfrazerlock.com/More_on_Linear_Models_Rcode1.r

Example: Body Fat

- Recall $p(\sigma^2 \mid \mathbf{y}) = IG\left(\frac{91}{2}, \frac{91\, s^2}{2}\right)$:



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Variance estimate, uninformative priors

- Note for the uninformative prior $p(\mu, \sigma^2) = \frac{1}{\sigma^2}$,

$Z \sim IG(a, b)$

$E(Z) = \frac{b}{a-1}$

$$E(\sigma^2 \mid \mathbf{y}) = \frac{s^2(n-p)}{n-p-2}$$

- However, the expected precision is

$$E(1/\sigma^2 \mid \mathbf{y}) = \frac{1}{s^2}$$

- s^2 still commonly used as point estimate for error variance.

Residuals

- Recall: defined Bayesian residual as

$$r_i' = y_i - E(Y_i \mid \mathbf{y}_{(i)})$$

where $\mathbf{y}_{(i)} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$

- For this context, the Bayesian residual is

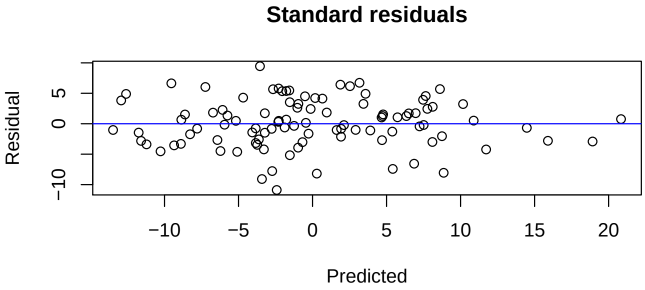
$$r_i' = y_i - \mathbf{x}_i \hat{\beta}_{(i)}$$

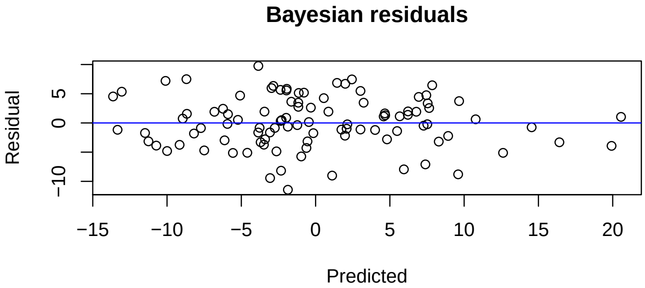
where $\hat{\beta}_{(i)} = (X_{(i)}^T X_{(i)})^{-1} X_{(i)}^T \mathbf{y}_{(i)}$.

- The standard (non-Bayesian) definition of residual is

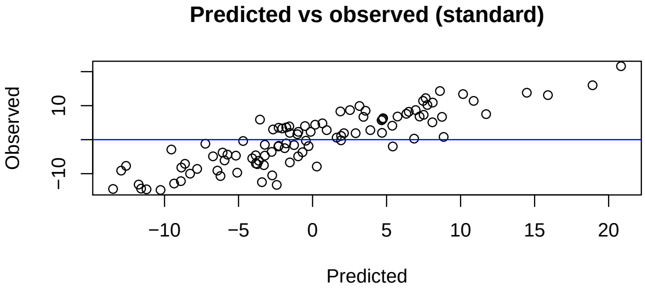
$$r_i = y_i - \mathbf{x}_i \hat{\beta}$$

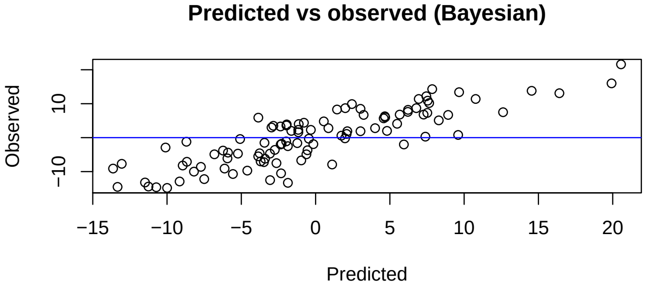
Example: Body Fat





Example: Body Fat





Normal-inverse-gamma prior

- Consider independent normal priors for the $\beta_i's$:

$$\beta \mid \sigma^2 \sim \text{Normal}(0, \sigma^2 T)$$

where $T_{ij} = \tau_i^2$ if $i = j$, 0 otherwise.

- And an inverse-gamma prior for σ^2:

$$\sigma^2 \sim IG(a, b).$$

- The full prior is

$$p(\beta, \sigma^2) = IG(\sigma^2 \mid a, b) \prod_{i=1}^{p} \text{Normal}(\beta_i \mid 0, \sigma^2 \tau_i^2)$$

Normal-inverse-gamma prior

- The posterior for β, given σ^2, is

$$p(\beta \mid \mathbf{y}, \sigma^2) = \text{Normal}\left(\tilde{\beta}, \sigma^2 V_\beta\right)$$

where $\tilde{\beta} = (X^T X + T^{-1})^{-1}(X^T \mathbf{y})$
and $V_\beta = (X^T X + T^{-1})^{-1}$

Normal-inverse-gamma prior

- The estimate $\tilde{\beta}$ solves a penalized least squares criterion:

$$\tilde{\beta} = \underset{\beta}{\text{argmin}} ||\mathbf{y} - XB||^2 + \sum_{i=1}^{p} \beta_i^2/\tau_i^2$$

$$= (y - X\beta)^T (y - X\beta) + \beta^T T^{-1} \beta$$

$$= y^T y - \beta^T X^T y - \beta^T X^T y + \beta^T X^T X \beta + \beta^T T^{-1} \beta$$

$$\frac{d}{d\beta} L = -2X^T y + 2X^T X \beta + 2T^{-1} \beta = 0$$

$$\longrightarrow X^T y = (X^T X + T^{-1}) \beta \longrightarrow \beta = (X^T X + T^{-1})^{-1} X^T y$$

- Shrinks unbiased estimate $\hat{\beta}$ toward $\mathbf{0}$.

Normal-inverse-gamma prior

- The marginal posterior for σ^2 is

$$p(\sigma^2 \mid \mathbf{y}) = IG\left(a_n, b_n\right)$$

where $a_n = a + \frac{n}{2}$ and $b_n = b + \frac{1}{2}[\mathbf{y}^T\mathbf{y} - \tilde{\beta}^T V_\beta^{-1}\tilde{\beta}]$

- The marginal posterior for β is a multivariate t-distribution

$$\frac{\beta_i - \tilde{\beta}_i}{\sqrt{\frac{b_n}{a_n}(V_\beta)_{ii}}} \sim t_{2a+n}.$$

- For a new predictor vector $\mathbf{x}_{n+1}$, the posterior predictive for y_{n+1} given σ^2 is

$$y_{n+1} \mid (\sigma^2, \vec{y}) \sim \text{Normal}(\mathbf{x}_{n+1}\tilde{\beta}, \sigma^2(1 + \mathbf{x}_{n+1} V_\beta \mathbf{x}_{n+1}^T))$$

$$y_{n+1} = x_{n+1}\beta + e_{n+1}$$

- The full posterior predictive distribution is a non-central t:

$$\frac{y_{n+1} - \mathbf{x}_{n+1}\tilde{\beta}}{\sqrt{\frac{b_n}{a_n}\left(1 + \mathbf{x}_{n+1} V_\beta \mathbf{x}_{n+1}^T\right)}} \sim t_{2a+n}.$$

$$E(y_{n+1} \mid \vec{y}, \sigma^2) = x_{n+1} E(\beta \mid \vec{y}, \sigma^2) + 0 = x_{n+1}\tilde{\beta}$$

$$V(y_{n+1} \mid \vec{y}, \sigma^2) = V(x_{n+1}\beta \mid \vec{y}, \sigma^2) + \sigma^2 = x_{n+1} V(\beta \mid \vec{y}, \sigma^2) x_{n+1}^T + \sigma^2$$

Extensions

- There are many other versions of the Bayesian linear model.
- E.g.: Could use non-trivial mean and covariance for β:

$$\beta \sim \text{Normal}(\mu_\beta, T)$$

- E.g.: Could relax iid assumption for $y_i's$, model general covariance:

$$\mathbf{y} \sim \text{Normal}(X\beta, \Sigma)$$

requires a prior for Σ.

- For more details and derivations see http://www.ericfrazerlock.com/LM_GoryDetails.pdf and Carlin & Louis 4.1.1