

Homework 2

1. The loss function is $l(\theta, a) = |\theta - a|$. Let m be the posterior median. We assume that $m < a$. Then,

$$|\theta - m| - |\theta - a| = \begin{cases} m - a & \text{if } \theta < m \\ 2\theta - m - a & \text{if } m \leq \theta < a \\ a - m & \text{if } \theta \geq a \end{cases}$$

$$\leq \begin{cases} m - a & \text{if } \theta \leq m \\ a - m & \text{if } \theta > a \end{cases}$$

Then,

$$\begin{aligned} E_{\theta|y}l(\theta|m) - E_{\theta|y}l(\theta|a) &= \int_{-\infty}^{\infty} \{|\theta - m| - |\theta - a|\} p(\theta|y) d\theta \\ &\leq \int_{-\infty}^m (m - a) p(\theta|y) d\theta + \int_m^{\infty} (a - m) p(\theta|y) d\theta \\ &= \frac{1}{2}(m - a) + \frac{1}{2}(a - m) \quad \left(\because \int_{-\infty}^m p(\theta|y) = \int_m^{\infty} p(\theta|y) = \frac{1}{2} \right) \\ &= 0 \end{aligned}$$

$$\therefore E_{\theta|y}l(\theta|m) \leq E_{\theta|y}l(\theta|a)$$

Similar for $a \leq m$.

2. a.) $[0, 1]$
 b.) $(y + 1)/(n + 2)$
 c.)

$$\begin{aligned} E\left(\frac{y+1}{n+2} - \theta\right)^2 &= E\left(\frac{(y+1)^2}{(n+2)^2} - 2\frac{y+1}{n+2}\theta + \theta^2\right) \\ &= \frac{1}{(n+2)^2}E(y^2) + \left(\frac{2}{(n+2)^2} - \frac{2}{n+2}\theta\right)E(y) + \theta^2 - \frac{2\theta}{n+2} + \frac{1}{(n+2)^2} \\ &= \frac{(4-n)\theta(\theta-1) + 1}{(n+2)^2} \end{aligned}$$

$$\therefore E(y^2) = \text{var}(y) + \{E(y)\}^2 = n\theta(1-\theta) + n^2\theta^2 \quad \text{and} \quad E(y) = n\theta$$

d.)

$$\begin{aligned} R(\theta, \hat{\theta}_{mle}) &< R(\theta, \hat{\theta}_{Bayes}) \\ \frac{\theta(1-\theta)}{n} &< \frac{(4-n)\theta(\theta-1) + 1}{(n+2)^2} \end{aligned}$$

Thus,

$$\theta \in \left(0, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{n+1}{2n+1}}\right) \quad \text{or} \quad \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{n+1}{2n+1}}, 1\right).$$

e.)

$$E\left(\frac{y}{n}\right) - \theta = 0$$

f.)

$$\begin{aligned} E\left(\frac{y+1}{n+2}\right) - \theta &= \frac{n\theta + 1}{n+2} - \theta \\ &= \frac{1 - 2\theta}{n+2} \end{aligned}$$

3. Let $d(y)$ satisfy $R(\theta, d(y)) = c$. Assume $d(y)$ is not a minimax rule. Then, there exists $d^*(y)$ that satisfies $\sup R(\theta, d^*(y)) < \sup R(\theta, d(y)) = c$.

$$r(\pi, d^*(y)) = \int R(\theta, d^*(y))\pi(\theta)d\theta < \int c \cdot \pi(\theta)d\theta = r(\pi, d(y))$$

This contradicts with bayes rule $d(y) = \arg \min_{d(y) \in \mathcal{D}} r(\pi, d(y))$.
 $\therefore d(y)$ is minimax.

4.

$$\begin{aligned} R(\mu, d_B) &= (1 - B)^2 \frac{\sigma^2}{n} + B^2 (\mu - \mu_0)^2 \\ r(\pi, d_B) &= E_\mu R(\mu, d_B) \\ &= (1 - B)^2 \frac{\sigma^2}{n} + B^2 E_\mu (\mu - \mu_0)^2 \\ &= \left(\frac{\sigma^2}{n} + \tau^2 \right) b^2 - \frac{2\sigma^2}{n} B + \frac{\sigma^2}{n} \\ &= \left(\frac{\sigma^2 + n\tau^2}{n} \right) \left(B - \frac{\sigma^2/n}{(\sigma^2 + n\tau^2)/n} \right)^2 + C \quad \text{find minimum by completing the square} \\ \therefore B &= \frac{\sigma^2}{\sigma^2 + n\tau^2} \end{aligned}$$

5. a.) [0.3920.823]
 b.) [0.3830.816]
 c.) The two intervals are equivalent when $y=8$. This creates a situation of Beta(9,9) which is perfectly symmetric.