More on Interval Estimation

PUBH 8442: Bayes Decision Theory and Data Analysis

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02/10/2021

Example: Emergency room (cont.)

- Consider the posterior predictive distribution of y_2 given y_1
  - y_1 is observed number of collapsed lung patients for initial week
  - y_2 is number of collapsed lung patients the following week

$y_1, y_2 \overset{iid}{\sim} \text{Pois}(\lambda)$

- Recall that under Jeffreys prior for λ,

$$p(\lambda \mid y_1) = \text{Gamma}(y_1 + \frac{1}{2}, 1) = \frac{\lambda^{y_1 - 1/2} e^{-\lambda}}{\Gamma(y_1 + \frac{1}{2})}$$

and

$$P(y_2 \mid \lambda) = \text{Poisson}(\lambda) = \frac{\lambda^{y_2} e^{-\lambda}}{y_2!}$$

Example: Emergency room (cont.)

- Posterior predictive is

$$p(y_2 \mid y_1) = \text{NegativeBinomial}(y_1 + 1/2, 1/2).$$

NegativeBinomial(r,p) has pmf $\frac{\Gamma(y+r)}{\Gamma(r)y!} p^r (1-p)^y$ for $y = 0, 1, 2, \ldots$

$$P(y_2 / y_1) = \int_0^\infty p(y_2 \mid \lambda) \cdot p(\lambda \mid y_1)\, d\lambda$$

$$= \int_0^\infty \frac{\lambda^{y_2} e^{-\lambda}}{y_2!} \cdot \frac{\lambda^{y_1 - \frac{1}{2}} e^{-\lambda}}{\Gamma(y_1 + \frac{1}{2})}\, d\lambda$$

$$\cdots = \frac{1}{y_2!\,\Gamma\left(y_1+\frac{1}{2}\right)} \int_0^\infty \underbrace{\lambda^{y_1+y_2-\frac{1}{2}} e^{-2\lambda}}_{\propto \, Gam\left(y_1+y_2+\frac{1}{2},\, 2\right)} d\lambda$$

$$= \frac{\Gamma\left(y_1+y_2+\frac{1}{2}\right)}{2^{y_1+y_2+\frac{1}{2}}}$$

$$= \frac{\Gamma\left(y_1+y_2+\frac{1}{2}\right)}{y_2!\,\Gamma\left(y_1+\frac{1}{2}\right)} \left(\frac{1}{2}\right)^{y_1+1} \left(\frac{1}{2}\right)^{y_2}$$

Example: Emergency room (cont.)

- Consider creating credible set for y_2, given y_1
- Action space is

$$\mathcal{A} = \{a : a \subseteq \{0, 1, 2, \ldots\}\}$$

- Loss function $l(y_2, d(y_1)) = \mathbb{1}_{\{y_2 \notin d(y_1)\}} + K \cdot |d(y_1)|$
  - $|\cdot|$ gives cardinality (number of values in $d(y_1)$)
  - Motivates $d(y_1) = \{k : P(y_2 = k \mid y_1) \geq K)$
  - Discrete analogue of HPD set, with level

$$1 - \alpha = \sum_{k \in d(y_1)} P(y_2 = k \mid y_1).$$

Example: Emergency room (cont.)

k	$P(y_2 = k \mid y_1 = 4)$
0	0.04
1	**0.10**
2	**0.14**
3	**0.15**
4	**0.14**
5	**0.12**
6	**0.09**
7	**0.07**
8	**0.05**
9	0.03

http://www.ericfrazerlock.com/More_on_Interval_estimation_Rcode2.r

- For $y_1 = 4$, $d(y_1) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ corresponds to $K = 0.05$ and

$$1-\alpha = 0.10+0.14+0.15+0.14+0.12+0.09+0.07+0.05 = 0.86.$$

Example: Emergency room (side note)

- Given y_1, consider the probability of no patients in following two weeks (A): $y_1, y_2, y_3 \overset{iid}{\sim} \text{Pois}(\lambda)$

$$P(A \mid y_1) = (1/3)^{y_1+1/2}$$

$$P(y_2=0 \cap y_3=0 \mid y_1) \cancel{= P(y_2=0 \mid y_1) \cdot P(y_3=0 \mid y_1)}$$

$$= P(y_2=0 \mid y_1) \cdot P(y_3=0 \mid y_1, y_2=0)$$

Note: $P(\lambda \mid y_2=0, y_1) = \text{Gamma}(y_1+\frac{1}{2}, 2)$

$\hookrightarrow P(y_3 \mid y_2=0, y_1) \sim NB(y_1+\frac{1}{2}, \frac{2}{3})$

$$P(y_2=0 \mid y_1) \cdot P(y_3=0 \mid y_2=0, y_1)$$

$$= \left(\frac{1}{2}\right)^{y_1+\frac{1}{2}} \left(\frac{2}{3}\right)^{y_1+\frac{1}{2}}$$

$$= \left(\frac{1}{3}\right)^{y_1+\frac{1}{2}}$$

Example: Emergency room (side note)

- Recall: only unbiased estimate for $P(A)$ is $(-1)^{y_1}$
- Note $E_{y_1 \mid \lambda}(1/3)^{y_1+1/2} = \frac{1}{\sqrt{3}}e^{-2\lambda/3}$
- So, our posterior predictive estimate has bias

$$\frac{1}{\sqrt{3}}e^{-2\lambda/3} - e^{-2\lambda}$$

$$= \sum_{k=0}^{\infty} (1/3)^{k+\frac{1}{2}} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\left(e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right)$$

$$= \sqrt{\tfrac{1}{3}}\, e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda/3)^k}{k!}$$

$$= \sqrt{\tfrac{1}{3}}\, e^{-\lambda} e^{\frac{\lambda}{3}} = \sqrt{\tfrac{1}{3}}\, e^{-\frac{2\lambda}{3}}$$

Normal model with Jeffreys priors

- Let $y_1, \ldots, y_n$ be iid $N(\mu, \sigma^2)$ with σ^2 known.
- The Jeffreys prior for μ is uniform: $p(\mu) \propto c$

$$\log p(\bar{y} \mid \mu) = C - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\frac{d}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \mu)$$

$$\frac{d^2}{d^2\mu} = -\frac{n}{\sigma^2}, \qquad I(\mu) = -E\left(-\frac{n}{\sigma^2}\right) = \frac{n}{\sigma^2}$$

$$p(\mu) \propto \sqrt{\frac{n}{\sigma^2}} \propto C$$

Normal model with Jeffreys priors

- $p(\mu) = c$ gives posterior $p(\mu \mid \mathbf{y}, \sigma^2) = \text{Normal}(\bar{y}, \frac{\sigma^2}{n})$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right\}$$

$$\propto \ldots \propto \exp\left\{-\frac{(\mu-\bar{y})^2}{2\sigma^2/n}\right\}$$

- So, the $100(1-\alpha)\%$ HPD (and quantile/symmetric) credible interval for μ is

$$C = \left(\bar{y} - \frac{\sigma z(\alpha/2)}{\sqrt{n}}, \bar{y} + \frac{\sigma z(\alpha/2)}{\sqrt{n}}\right)$$

where $z(\alpha/2)$ is the $\alpha/2$ quantile of Normal$(0,1)$.

Normal model with Jeffreys priors

- Now assume σ^2 is also unknown, and independent of μ
- The Jeffreys prior for σ^2 is $p(\sigma^2) \propto \frac{1}{\sigma^2}$
- Then,
$$p(\mu \mid \mathbf{y}) = \int_0^\infty p(\mu \mid \mathbf{y}, \sigma^2) p(\sigma^2 \mid \mathbf{y})\, d\sigma^2$$
is a shifted and scaled t-distribution:
$$\frac{\mu - \bar{y}}{s/\sqrt{n}} \sim t_{n-1}$$
- The HPD credible interval for μ is a standard t-interval with $n-1$ degrees of freedom.