More on Priors

PUBH 8442: Bayes Decision Theory and Data Analysis

Eric F. Lock
UMN Division of Biostatistics, SPH
elock@umn.edu

01/29/2025

Non-informative priors

- A non-informative prior is intended to convey no prior belief
  - Let the data alone drive inference.
- Example: For $\Theta = \{\theta_1, \ldots, \theta_K\}$

$$P(\theta = \theta_k) = \frac{1}{K}.$$

- Example: For $\Theta = [a, b]$,

$$p(\theta) = \frac{1}{b - a} \quad \text{for} \;\; a \leq \theta \leq b.$$

- Note: If $p(\theta) = c \;\forall\; \theta \in \Theta$, then

$$p(\theta \mid \mathbf{y}) \propto p(\mathbf{y} \mid \theta),$$

and so the mode of $p(\theta \mid \mathbf{y})$ is the MLE for θ.

Improper priors

- A "prior" that is not a true probability distribution is *improper*
  - An improper prior can result in an improper posterior
- Example: $p(\theta) = 1$ for $\theta \in (-\infty, \infty)$.
  - Improper because $\int p(\theta)\, d\theta = \infty$
  - If $\int p(\mathbf{y} \mid \theta)\, d\theta < \infty$,

$$p(\theta \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \theta)}{\int p(\mathbf{y} \mid \theta)\, d\theta}$$

    is still proper.
- Improper priors should be avoided if possible.
  - But can give reasonable results, especially if posterior is proper.

Example: Venusian surface

- A spacecraft lands on the planet Venus, and captures a small surface sample.
- The radioactivity of the sample is measured, as the frequency of particle emission.
- $y = 60$ particle emissions are observed in one minute.
- Then the spacecraft dies, due to high surface temperature and pressure.
- Infer θ, the average particle emissions per minute.

$$y \sim \text{Poisson}(\theta).$$

Example: Venusian surface

- Consider the uniform prior

$$p(\theta) = c \text{ for } \theta > 0.$$

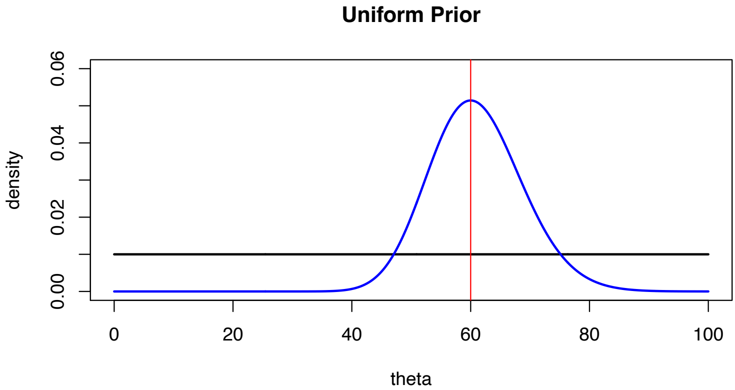
  - NOTE: $p(\theta)$ is improper.

- Posterior IS proper:

$$p(\theta \mid y = 60) = \text{Gamma}(61, 1)$$

Example: Venusian surface

- **Prior** and posterior densities, given $y = 60$:



Code:

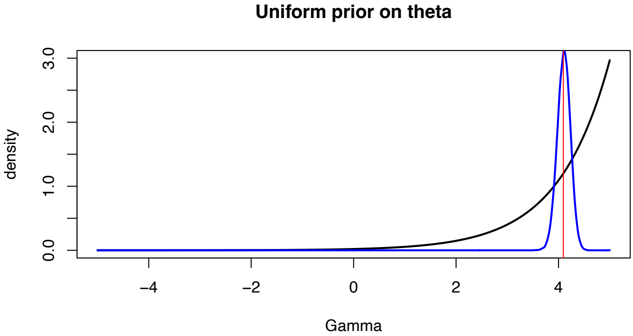
http://www.ericfrazerlock.com/More_on_priors_Rcode1.r

Example: Venusian surface

- Radioactivity by emission frequency is often given on log scale:

$$\gamma = \log(\theta).$$

- A uniform prior for θ implies prior $p^*(\gamma) \propto e^\gamma$ for γ:



-

- Prior is not "non-informative" for γ!

Jeffreys priors

- For a one-parameter model $p(\mathbf{y} \mid \theta)$, the Fisher information is

$$I(\theta) = -E_{\mathbf{y} \mid \theta} \left[\frac{d^2}{d\theta^2} \log p(\mathbf{y} \mid \theta) \right].$$

- The Jeffreys prior is proportional to the square root of I:

$$p(\theta) \propto [I(\theta)]^{1/2}$$

- Jeffreys priors are invariant under all one-to-one transformations:
  - If $\gamma = h(\theta)$ and $p(\theta)$ is Jeffreys for θ, then the induced prior for γ is Jeffreys for γ.
- Satisfy other "non-informative" properties (Box & Tiao, Sec 1.3).

Jeffreys priors

- For multi-parameter $\theta = (\theta_1, \ldots, \theta_K)$, the Fisher information is a matrix with i, j element

$$I_{ij}(\theta) = -E_{\mathbf{y} \mid \theta}\left[\frac{d^2}{d\theta_i\theta_j} \log p(\mathbf{y} \mid \theta)\right].$$

- And the Jeffreys prior is

$$p(\theta) \propto [\det I(\theta)]^{1/2}$$

Jeffreys priors

- For a Poisson(θ) model, the Jeffreys prior for θ is

$$p(\theta) \propto \frac{1}{\sqrt{\theta}}$$

- For a Binomial(n, θ) model, the Jeffreys prior for θ is

$$p(\theta) = \text{Beta}(0.5, 0.5)$$

  - Homework

Elicited priors

- Often desire an expert to elicit the prior for a given experiment
- Issues:
  - Human intuition generally approximates uncertainty poorly
  - Non-statistician experts have little knowledge of probability distributions
- Have expert create a probability histogram
- Or specify certain desirable properties (e.g., quantiles) and choose a known distribution to match those properties.
- Prior elicitation software: `https://shelf.sites.sheffield.ac.uk/`.

Poisson-gamma model

- Let $y \sim \text{Poisson}(\theta)$ and $\theta \sim \text{Gamma}(\alpha, \beta)$:
$$p(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}.$$
- Then, $p(\theta \mid y) = \text{Gamma}(\alpha + y, \beta + 1)$

Example: Venusian surface (cont.)

- An expert in planetary radioactivity is specifies percentiles for radioactivity of Venusian surface
  - 10th percentile: 1, 90th percentile: 80
  - "80 percent sure that between 1 and 80 parts will emit per minute"
- Fit a Gamma prior using this information.
  - Code: http://www.ericfrazerlock.com/More_on_priors_Rcode2.r
  - $\theta \sim \text{Gamma}(0.62, 0.02)$
- Posterior after $y = 60$:

$$\text{Gamma}(60.62, 1.02)$$

Example: Venusian surface (cont.)

- **Prior** and posterior densities, given $y = 60$: