Homework 1

1. a.) Using the formula for conditional probability,

$$P(\text{Free if Spam}) = \frac{P(\text{Free and Spam})}{P(\text{Spam})} = \frac{0.0357}{0.134} = 0.266$$

b.) Using the formula for conditional probability,

$$P(\text{Spam if Free}) = \frac{P(\text{Free and Spam})}{P(\text{Free})} = \frac{0.0357}{0.0475} = 0.752$$

c.) Using the Bayes' rule,

$$P(\text{Spam if Text}) = \frac{P(\text{Spam})P(\text{Text if Spam})}{P(\text{Text})} = \frac{0.134 \cdot 0.3855}{0.0701} = 0.737$$

d.) Using Bayes' rule,

$$\begin{aligned} P(\text{Spam if Text and Free}) &= \frac{P(\text{Spam})P(\text{Text and Free if Spam})}{P(\text{Spam})P(\text{Text and Free if Spam}) + P(\text{not Spam})P(\text{Text and Free if not Spam})} \\ &= \frac{0.134 \cdot 0.17}{0.134 \cdot 0.17 + 0.866 \cdot 0.0006} \\ &= 0.978. \end{aligned}$$

e.) Applying the total probability rule,

$$\begin{aligned} P(\text{Free and Text}) &= P(\text{Spam})P(\text{Text and Free if Spam}) + P(\text{not Spam})P(\text{Text and Free if not Spam}) \\ &= 0.134 \cdot 0.17 + 0.866 \cdot 0.0006 \\ &= 0.0233. \end{aligned}$$

Again applying the total probability rule,

$$P(\text{Free and not Text}) = P(\text{Free}) - P(\text{Free and Text}) = 0.0475 - 0.0233 = 0.0242$$

and

$$P(\text{Free and not Text if Spam}) = P(\text{Free if Spam}) - P(\text{Free and Text if Spam}) = 0.2664 - 0.1700 = 0.0964.$$

So, applying Bayes' rule,

$$P(\text{Spam if Free and not Text}) = \frac{P(\text{Spam})P(\text{Free and not Text if Spam})}{P(\text{Free and not Text})} = \frac{0.134 \cdot 0.0964}{0.0242} = 0.534.$$

2. Let B and R denote the position of the blue and red arrow, respectively. At i^{th} spin of the red arrow (note that B is fixed at unknown b), we observe $X_i = 1$ if $R_i > b$, $X_i = 0$ o.w. (Binomial distribution).

a.) The position of the blue arrow has a uniform distribution. Thus, the probability density is updated as a beta-binomial model.

$$\begin{aligned} X_i|B = b &\sim Bin(H + L, b) \\ b &\sim \text{Beta}(1, 1) \end{aligned}$$

$$p(b|x) \propto p(b) \prod_{i}^{H+L} f(X_i|B = b) = b^L(1 - b)^H$$

$$b|x \sim \text{Beta}(L + 1, H + 1)$$

b.) The mean of $b|x$ is $\frac{L+1}{L+H+2}$.

3. Note that $\mu \sim N(\mu_0, \tau^2)$ and $y_i|\mu \sim N(\mu, \sigma^2)$. Then

$$\begin{aligned}
p(\mu|\mathbf{y}) &\propto p(\mu)p(\mathbf{y}|\mu) \\
&\propto \exp\{-\frac{(\mu-\mu_0)^2}{2\tau^2} - \frac{-\sum(y_i-\mu)^2}{2\sigma^2}\} \\
&\propto \exp\{-\frac{\mu^2 - 2\mu\mu_0}{2\tau^2} - \frac{-2n\bar{y}\mu + n\mu^2}{2\sigma^2}\} \\
&\propto \exp\{-\frac{\mu^2 - 2\mu\left(\frac{\mu_0\sigma^2+\bar{y}n\tau^2}{\sigma^2+n\tau^2}\right)}{2\left(\frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}\right)}\} \\
&\sim N\left(\frac{\mu_0\sigma^2+\bar{y}n\tau^2}{\sigma^2+n\tau^2}, \frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}\right)
\end{aligned}$$

4. Using a trick to replace densities by "models",

$$\begin{aligned}
y_i &= \mu + \epsilon_1, \epsilon \sim N(0, \sigma^2) \\
\mu &= \mu_0 + \epsilon_2, \epsilon \sim N(0, \tau^2) \\
\rightarrow y_i &= \mu_0 + \epsilon_1 + \epsilon_2 \\
&= \mu_0 + \epsilon^*, \epsilon^* \sim N(0, \sigma^2 + \tau^2)
\end{aligned}$$

Thus, the prior predictive distribution $y_i \sim N(\mu_0, \sigma^2 + \tau^2)$. Similarly,

$$\begin{aligned}
y_{n+1} &= \mu + \epsilon_1, \epsilon_1 \sim N(0, \sigma^2) \\
\mu &= \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2+n\tau^2} + \epsilon_3, \epsilon_3 \sim N(0, \frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}) \\
\rightarrow y_{n+1} &= \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2+n\tau^2} + \epsilon_1 + \epsilon_3 \\
y_{n+1} &= \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2+n\tau^2} + \epsilon^{**}, \epsilon^{**} \sim N(0, \frac{\sigma^2\tau^2}{\sigma^2+n\tau^2} + \sigma^2)
\end{aligned}$$

Thus, the posterior predictive distribution $y_{n+1}|\mathbf{y} \sim N(\frac{\sigma^2\mu_0+n\tau^2\bar{y}}{\sigma^2+n\tau^2}, \frac{\sigma^2\tau^2}{\sigma^2+n\tau^2} + \sigma^2)$

5. The Fisher information is

$$\begin{aligned}
I(p) &= -E_{y|p}\left[\frac{d^2}{dp^2}\log f(y|p)\right] \\
&= -E_{y|p}\left[\frac{d^2}{dp^2}\log p^y(1-p)^{1-y}\right] \\
&= -E_{y|p}\left[-\frac{y}{p^2} - \frac{1-y}{(1-p)^2}\right] \\
&= \frac{1}{p} + \frac{1}{1-p}
\end{aligned}$$

Then, the Jeffreys prior is

$$\pi(p) \propto \sqrt{\frac{1}{p} + \frac{1}{1-p}} = p^{\frac{1}{2}-1}(1-p)^{\frac{1}{2}-1}$$

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