Homework 2

1. The loss function is $l(\theta, a) = |\theta - a|$. Let m be the posterior median. We assume that $m < a$. Then,

$$|\theta - m| - |\theta - a| = \begin{cases} m - a & \text{if} \quad \theta < m \\ 2\theta - m - a & \text{if} \quad m \le \theta < a \\ a - m & \text{if} \quad \theta \ge a \end{cases}$$
$$\le \begin{cases} m - a & \text{if} \quad \theta \le m \\ a - m & \text{if} \quad \theta > a \end{cases}$$

Then,

$$\begin{aligned} E_{\theta|y} l(\theta|m) - E_{\theta|y} l(\theta|a) &= \int_{-\infty}^{\infty} \{|\theta - m| - |\theta - a|\} p(\theta|y) \mathrm{d}\theta \\ &\le \int_{-\infty}^{m} (m - a) p(\theta|y) \mathrm{d}\theta + \int_{m}^{\infty} (a - m) p(\theta|y) \mathrm{d}\theta \\ &= \frac{1}{2}(m - a) + \frac{1}{2}(a - m) \quad \left(\because \int_{-\infty}^{m} p(\theta|y) = \int_{m}^{\infty} p(\theta|y) = \frac{1}{2} \right) \\ &= 0 \end{aligned}$$

$$\therefore E_{\theta|y} l(\theta|m) \le E_{\theta|y} l(\theta|a)$$

Similar for $a \le m$.

2. a.) $[0, 1]$

   b.) $(y + 1)/(n + 2)$

   c.)

$$\begin{aligned} E\left(\frac{y+1}{n+2} - \theta\right)^2 &= E\left(\frac{(y+1)^2}{(n+2)^2} - 2\frac{y+1}{n+2}\theta + \theta^2\right) \\ &= \frac{1}{(n+2)^2} E(y^2) + \left(\frac{2}{(n+2)^2} - \frac{2}{n+2}\theta\right) E(y) + \theta^2 - \frac{2\theta}{n+2} + \frac{1}{(n+2)^2} \\ &= \frac{(4-n)\theta(\theta-1)+1}{(n+2)^2} \end{aligned}$$

$$\because E(y^2) = var(y) + \{E(y)\}^2 = n\theta(1-\theta) + n^2\theta^2 \quad \text{and} \quad E(y) = n\theta$$

   d.)

$$\frac{\theta(1-\theta)}{n}$$

   e.)

$$\begin{aligned} R(\theta, \hat{\theta}_{mle}) &< R(\theta, \hat{\theta}_{Bayes}) \\ \frac{\theta(1-\theta)}{n} &< \frac{(4-n)\theta(\theta-1)+1}{(n+2)^2} \end{aligned}$$

   Thus,

$$\theta \in (0, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{n+1}{2n+1}}) \quad \text{or} \quad (\frac{1}{2} + \frac{1}{2}\sqrt{\frac{n+1}{2n+1}}).$$

   f.)

$$E(\frac{y}{n}) - \theta = 0$$

g.)

$$E(\frac{y+1}{n+2}) - \theta = \frac{n\theta+1}{n+2} - \theta$$
$$= \frac{1-2\theta}{n+2}$$

3. Let $d(y)$ satisfy $R(\theta, d(y)) = c$. Assume $d(y)$ is not a minimax rule. Then, there exists $d^*(y)$ that satisfies $\sup R(\theta, d^*(y)) < \sup R(\theta, d(y)) = c$.

$$r(\pi, d^*(y)) = \int R(\theta, d^*(y))\pi(\theta)\mathrm{d}\theta < \int c \cdot \pi(\theta)\mathrm{d}\theta = r(\pi, d(y))$$

This contradicts with bayes rule $d(y) = \arg\min_{d(y)\in\mathcal{D}} r(\pi, d(y))$.
$\therefore d(y)$ is minimax.

4.

$$\begin{aligned}
R(\mu, d_B) &= (1-B)^2\frac{\sigma^2}{n} + B^2(\mu-\mu_0)^2 \\
r(\pi, d_B) &= E_\mu R(\mu, d_B) \\
&= (1-B)^2\frac{\sigma^2}{n} + B^2 E_\mu(\mu-\mu_0)^2 \\
&= \left(\frac{\sigma^2}{n} + \tau^2\right) b^2 - \frac{2\sigma^2}{n}B + \frac{\sigma^2}{n} \\
&= \left(\frac{\sigma^2 + n\tau^2}{n}\right)\left(B - \frac{\sigma^2/n}{(\sigma^2+n\tau^2)/n}\right)^2 + C \quad \text{find minimum by completing the square}
\end{aligned}$$

$$\therefore B = \frac{\sigma^2}{\sigma^2 + n\tau^2}$$

5. a.) $[0.467, 0.894]$
   b.) $[0.449, 0.882]$
   c.) The two intervals are equivalent when y=7. This creates a situation of Beta(8,8) which is perfectly symmetric.