Homework 3

1. The loss function is given by

$$\begin{aligned} I(\mu, \mathrm{R}) &= 500 \\ I(\mu, \mathrm{N}) &= 1000 \cdot I(\mu < 11.9). \end{aligned}$$

Then, the posterior loss is

$$\begin{aligned} \rho(\pi, \mathrm{R}) &= 500 \\ \rho(\pi, \mathrm{N}) &= 1000 \cdot p(\mu < 11.9|\mathbf{y}). \end{aligned}$$

Note that the posterior distribution

$$p(\mu|\mathbf{y}) \sim N\left(\frac{12 \cdot 0.05 + 0.01n\bar{y}}{0.05 + 0.01n}, \frac{0.05 \cdot 0.01}{0.05 + 0.01n}\right).$$

Since $500 = \rho(\pi, \mathrm{R}) < \rho(\pi, \mathrm{N}) = 1000 \cdot p(\mu < 11.9|\mathbf{y})$ when $\rho(\mu < 11.9|\mathbf{y}) > 0.5$, the Bayes decision rule is to

$$\begin{aligned} &\text{Recalibrate} \quad \text{if} \quad \bar{y} < \frac{11.9(0.05 + 0.01n) - 0.6}{0.01n} \\ &\text{Not recalibrate} \quad \text{if} \quad \text{o.w.} \end{aligned}$$

2. a.) Note that the partial Bayes factor for M_1 over M_2 is

$$\mathrm{BF}(y_2|y_1) = \frac{p(y_2|y_1, M_1)}{p(y_2|y_1, M_2)} = \frac{\int p(y_2|\mu)p(\mu|y_1, M_1)\mathrm{d}\mu}{\int p(y_2|\mu)p(\mu|y_1, M_2)\mathrm{d}\mu}.$$

Both the numerator and the denominator are posterior predictive. Thus, we derive

$$\begin{aligned} \mu|y_1, M_1 &\sim N\left(\frac{\tau^2 y_1 + \sigma^2\mu_0}{\tau^2 + \sigma^2}, \frac{\tau^2\sigma^2}{\tau^2 + \sigma^2}\right) \\ y_2|y_1, M_1 &\sim N\left(\frac{\tau^2 y_1 + \sigma^2\mu_0}{\tau^2 + \sigma^2}, \frac{\tau^2\sigma^2}{\tau^2 + \sigma^2} + \sigma^2\right) \\ \mu|y_1, M_2 &\sim N\left(y_1, \sigma^2\right) \\ y_2|y_1, M_2 &\sim N\left(y_1, 2\sigma^2\right). \end{aligned}$$

Then,

$$\mathrm{BF}(y_2|y_1) = \left(2 \cdot \frac{\tau^2 + \sigma^2}{2\tau^2 + \sigma^2}\right)^{1/2} \exp\left[-\frac{1}{2}\left\{\left(\frac{\tau^2\sigma^2}{\tau^2 + \sigma^2} + \sigma^2\right)^{-1}\left(y_2 - \frac{\tau^2 y_1 + \sigma^2\mu_0}{\tau^2 + \sigma^2}\right)^2 - (2\sigma^2)^{-1}(y_2 - y_1)^2\right\}\right].$$

b.) The arithmetic intrinsic Bayes factor for M_1 over M_2 is

$$\frac{\mathrm{BF}(y_2|y_1) + \mathrm{BF}(y_1|y_2)}{2} = 0.715$$

Given the data (y_1, y_2), there is insufficient evidence to conclude that M_2 is more reasonable than M_1. There is insufficient evidence to conclude that this individual is an outlier.

3. a.)

$$
\begin{aligned}
p(y|M_1) &= \binom{n}{y}\left(\frac{1}{2}\right)^n \\
p(y|M_2) &= \int_{\infty}^{\infty} p(y|\theta)p(\theta|M_2)\mathrm{d}\theta \\
&= \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\mathrm{d}\theta \\
&= \binom{n}{y}\mathrm{B}(y+1, n-y+1)
\end{aligned}
$$

Then, the Bayes factor is

$$
\mathrm{BF} = \frac{p(y|M_1)}{p(y|M_2)} = \frac{\binom{n}{y}\left(\frac{1}{2}\right)^n}{\binom{n}{y}\mathrm{B}(y+1, n-y+1)} = \frac{(n+1)!}{2^n y!(n-y)!}.
$$

b.)

$$
\begin{aligned}
p(y|M_3) &= \int_{\infty}^{\infty} p(y|\theta)p(\theta|M_3)\mathrm{d}\theta \\
&= \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{1}{\mathrm{B}(\alpha,\alpha)}\theta^{\alpha-1}(1-\theta)^{\alpha-1}\mathrm{d}\theta \\
&= \frac{n!}{y!(n-y)!}\frac{\mathrm{B}(\alpha+y, \alpha+n-y)}{\mathrm{B}(\alpha,\alpha)}
\end{aligned}
$$

Then, the Bayes factor is

$$
\mathrm{BF}^* = \frac{p(y|M_3)}{p(y|M_2)} = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\cdot\frac{\Gamma(y+\alpha)\Gamma(n-y+\alpha)\Gamma(n+2)}{\Gamma(n+2\alpha)\Gamma(y+1)\Gamma(n-y+1)}
$$

c.) When $\alpha = 1$, $\mathrm{BF}^* = 1$.
When $\alpha \to \infty$,

$$
\begin{aligned}
\lim_{\alpha\to\infty}\mathrm{BF}^* &= \lim_{\alpha\to\infty}\frac{\Gamma(\alpha+y)\Gamma(\alpha+n-y)\Gamma(2\alpha)\Gamma(n+2)}{\Gamma(2\alpha+n)\Gamma(\alpha)\Gamma(\alpha)\Gamma(y+1)\Gamma(n-y+1)} \\
&= \lim_{\alpha\to\infty}\frac{\alpha^y\alpha^{(n-y)}\Gamma(2\alpha)\Gamma(n+2)}{\Gamma(2\alpha)(2\alpha)^n\Gamma(y+1)\Gamma(n-y+1)} \\
&= \frac{1}{2^n}\frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}
\end{aligned}
$$