

Homework 3

1. The loss function is given by

$$\begin{aligned} I(\mu, R) &= 500 \\ I(\mu, N) &= 1000 \cdot I(\mu < 11.9). \end{aligned}$$

Then, the posterior loss is

$$\begin{aligned} \rho(\pi, R) &= 500 \\ \rho(\pi, N) &= 1000 \cdot p(\mu < 11.9|\mathbf{y}). \end{aligned}$$

Note that the posterior distribution

$$p(\mu|\mathbf{y}) \sim N\left(\frac{12 \cdot 0.05 + 0.01n\bar{y}}{0.05 + 0.01n}, \frac{0.05 \cdot 0.01}{0.05 + 0.01n}\right).$$

Since $500 = \rho(\pi, R) < \rho(\pi, N) = 1000 \cdot p(\mu < 11.9|\mathbf{y})$ when $\rho(\mu < 11.9|\mathbf{y}) > 0.5$, the Bayes decision rule is to

$$\begin{aligned} \text{Recalibrate} &\quad \text{if } \bar{y} < \frac{11.9(0.05 + 0.01n) - 0.6}{0.01n} \\ \text{Not recalibrate} &\quad \text{if o.w.} \end{aligned}$$

2. a.) Note that the partial Bayes factor for M_1 over M_2 is

$$\text{BF}(y_2|y_1) = \frac{p(y_2|y_1, M_1)}{p(y_2|y_1, M_2)} = \frac{\int p(y_2|\mu)p(\mu|y_1, M_1)d\mu}{\int p(y_2|\mu)p(\mu|y_1, M_2)d\mu}.$$

Both the numerator and the denominator are posterior predictive. Thus, we derive

$$\begin{aligned} \mu|y_1, M_1 &\sim N\left(\frac{\tau^2 y_1 + \sigma^2 \mu_0}{\tau^2 + \sigma^2}, \frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2}\right) \\ y_2|y_1, M_1 &\sim N\left(\frac{\tau^2 y_1 + \sigma^2 \mu_0}{\tau^2 + \sigma^2}, \frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2} + \sigma^2\right) \\ \mu|y_1, M_2 &\sim N(y_1, \sigma^2) \\ y_2|y_1, M_2 &\sim N(y_1, 2\sigma^2). \end{aligned}$$

Then,

$$\text{BF}(y_2|y_1) = \left(2 \cdot \frac{\tau^2 + \sigma^2}{2\tau^2 + \sigma^2}\right)^{1/2} \exp\left[-\frac{1}{2} \left\{ \left(\frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2} + \sigma^2\right)^{-1} \left(y_2 - \frac{\tau^2 y_1 + \sigma^2 \mu_0}{\tau^2 + \sigma^2}\right)^2 - (2\sigma^2)^{-1} (y_2 - y_1)^2 \right\}\right].$$

b.) The arithmetic intrinsic Bayes factor for M_1 over M_2 is

$$\frac{\text{BF}(y_2|y_1) + \text{BF}(y_1|y_2)}{2} = 1.09$$

Given the data (y_1, y_2) , there is insufficient evidence to conclude that M_2 is more reasonable than M_1 . We do not have sufficient evidence to conclude that this observation is an outlier. It is most likely not an outlier.

3. a.)

$$\begin{aligned}
 p(y|M_1) &= \binom{n}{y} \left(\frac{1}{2}\right)^n \\
 p(y|M_2) &= \int_0^1 p(y|\theta)p(\theta|M_2)d\theta \\
 &= \int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} d\theta \\
 &= \binom{n}{y} B(y+1, n-y+1)
 \end{aligned}$$

Then, the Bayes factor is

$$\text{BF} = \frac{p(y|M_1)}{p(y|M_2)} = \frac{\binom{n}{y} \left(\frac{1}{2}\right)^n}{\binom{n}{y} B(y+1, n-y+1)} = \frac{(n+1)!}{2^n y!(n-y)!}$$

b.)

$$\begin{aligned}
 p(y|M_3) &= \int_0^1 p(y|\theta)p(\theta|M_3)d\theta \\
 &= \int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} \frac{1}{B(\alpha, \alpha)} \theta^{\alpha-1} (1-\theta)^{\alpha-1} d\theta \\
 &= \frac{n!}{y!(n-y)!} \frac{B(\alpha+y, \alpha+n-y)}{B(\alpha, \alpha)}
 \end{aligned}$$

Then, the Bayes factor is

$$\text{BF}^* = \frac{p(y|M_3)}{p(y|M_2)} = \frac{n!}{y!(n-y)!} \frac{B(\alpha+y, \alpha+n-y)}{B(\alpha, \alpha)} \frac{y!(n-y)!}{n!} \frac{1}{B(y+1, n-y+1)} = \frac{B(\alpha+y, \alpha+n-y)}{B(\alpha, \alpha)B(y+1, n-y+1)}$$

c.) When $\alpha = 1$, $\text{BF}^* = 1$. When $\alpha \rightarrow \infty$,

$$\begin{aligned}
 \lim_{\alpha \rightarrow \infty} \text{BF}^* &= \lim_{\alpha \rightarrow \infty} \frac{\Gamma(\alpha+y)\Gamma(\alpha+n-y)\Gamma(2\alpha)\Gamma(n+2)}{\Gamma(2\alpha+n)\Gamma(\alpha)\Gamma(\alpha)\Gamma(y+1)\Gamma(n-y+1)} \\
 &= \lim_{\alpha \rightarrow \infty} \frac{\alpha^n \Gamma(n+2)}{2^n \alpha^n \Gamma(y+1)\Gamma(n-y+1)} \\
 &= \frac{\Gamma(n+2)}{2^n \Gamma(y+1)\Gamma(n-y+1)} \\
 &= \frac{1}{2^n B(y+1, n-y+1)}
 \end{aligned}$$