Homework 4

Due Thursday March 14th by end of day, via Canvas

PUBH 8442: Bayes Decision Theory and Data Analysis

1. Under the normal-normal hierarchical model as described in the Hierarchical Models slide set, show that the posterior distribution for μ is

$$p(\mu \mid \mathbf{y}) = \text{Normal}(\hat{\mu}, V_\mu)$$

where

$$\hat{\mu} = \frac{\sum_{i=1}^m (\sigma_i^2 + \tau^2)^{-1} \bar{y}_i}{\sum_{i=1}^m (\sigma_i^2 + \tau^2)^{-1})} \text{ and } V_\mu = \left[\sum_{i=1}^m (\sigma_i^2 + \tau^2)^{-1} \right]^{-1}.$$

2. *Hierarchical regression model - known variance*

Let $\mathbf{y} = (\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_K')'$ be an $N \times 1$ vector of outcomes that has been partitioned into K blocks (or sub-vectors) such that $\mathbf{y}_k$ is $n_k \times 1$ and $\sum_{k=1}^K n_k = N$. Each $\mathbf{y}_k$ can be considered as n_k observations from group k. Suppose we assume that each block is independent and consider the following (hierarchical) model:

$$\mathbf{y}_k \sim \text{Normal}(X_k \beta_k, \sigma^2 I),$$

$$\beta_k \sim \text{Normal}(\beta_0, \sigma^2 \tau_k^2 I),$$

$$\beta_0 \sim \text{Normal}(\mathbf{0}, \sigma^2 \tau_0^2 I),$$

with the τ_k^2, τ_0^2, σ^2 assumed known. The full joint density is

$$p(\mathbf{y}, \beta_0, \beta_1, \ldots, \beta_K) = N(\beta_0 \mid \mathbf{0}, \sigma^2 \tau_0^2 I) \cdot \prod_{k=1}^K N(\beta_k \mid \beta_0, \sigma^2 \tau_k^2 I) \cdot \prod_{k=1}^K N(\mathbf{y}_k \mid X_k \beta_k, \sigma^2 I).$$

(a) Find the conditional posterior distribution of each β_k: $p(\beta_k \mid \mathbf{y}, \beta_0)$

(b) Find the conditional posterior distribution of β_0: $p(\beta_0 \mid \mathbf{y}, \{\beta_k\}_{k=1}^K)$

(c) Find the conditional predictive distribution for a single observation from a new group $k+1$ ($n_{k+1} = 1$) with observed covariates $\mathbf{x}_{k+1}$: $p(y_{k+1} \mid \mathbf{y}, \beta_0, \{\beta_k\}_{k=1}^K)$.