Midterm [20 pts]

March 16th, 2022 in class

PUBH 8442: Bayes Decision Theory and Data Analysis

Give your final answers in simplified, closed form wherever possible. However, partial credit will be awarded for incomplete solutions. Good luck!

1. *Regression hypotheses* [9 pts]
   Consider the simple linear regression model $\mathbf{y} = \mathbf{x}\beta + \epsilon$, where $\mathbf{y} : n \times 1$, $\mathbf{x} : n \times 1$ and $\epsilon : n \times 1$ with $\epsilon \sim N(\mathbf{0}, \sigma^2 I)$. Assume the residual variance σ^2 is fixed, and there are two models for β:

   - M_0 is the null model: $\beta = 0$.
   - M_1 is a model with a normal prior: $\beta \sim N(0, \sigma^2\tau^2)$ with τ^2 fixed.

   Assume each model has prior probability 1/2, $P(M_0) = P(M_1) = 1/2$.

   (a) (3 points) What is the marginal distribution for $\mathbf{y}$ under M_1, $p(\mathbf{y} \mid M_1)$?

   (b) (3 points) What is the posterior expectation of β, marginalizing over M_0 and M_1, $E(\beta \mid \mathbf{y})$?

2. *Telemedicine call-ins* [7 pts]
   Let Y be the number of call-ins to a new telemedicine service within the first hour, which has a Poisson distribution with rate parameter λ: $Y \sim \text{Poisson}(\lambda)$. Assume λ has a $\text{Gamma}(1, 1)$ prior distribution.

   (a) (3 pts) What is the marginal probability that $Y = 0$, $P(Y = 0)$?

   (b) (4 pts) Consider the wait time (in hrs) until the next call-in after the first hour, X, where $X \sim \text{Exponential}(\lambda)$. What is the conditional density of X, given Y, $p(X \mid Y)$? (Assume X and Y are independent given λ.)

3. *Bayes estimator* [4 pts]
   Consider a Bayesian model with likelihood $p(\mathbf{y} \mid \theta)$ and prior $p(\theta)$ defined for $\theta > 0$. We would like to estimate θ using the loss function

   $$l(\theta, d(\mathbf{y})) = \frac{(\theta - d(\mathbf{y}))^2}{d(\mathbf{y})} \quad \text{for } d(\mathbf{y}) > 0.$$

   What is the Bayes decision rule, $d(\mathbf{y}) = \hat{\theta}_{\mathbf{y}}$, under this loss function? Write your answer in terms of the posterior mean $\text{E}(\theta \mid \mathbf{y})$ and variance $\text{V}(\theta \mid \mathbf{y})$.

Reference: Density and probability mass functions

$\text{Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) : p(\mathbf{x}) = \frac{e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}}{\det(2\pi\Sigma)^{1/2}}$

$\text{Exponential}(\lambda) : p(x) = \lambda e^{-\lambda x}$

$\text{Gamma}(\alpha, \beta) : p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$\text{Poisson}(\lambda) : P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$