Midterm [20 pts]

March 19th, 2025 in class

PUBH 8442: Bayes Decision Theory and Data Analysis

Give your final answers in simplified, closed form wherever possible. However, partial credit will be awarded for incomplete solutions. Good luck!

1. *Support hotline* [11 pts]
   Consider incoming calls to a support hotline. The number of calls taken for a given day i, c_i, is Poisson distributed with rate parameter λ:

$$c_i \overset{\text{iid}}{\sim} \text{Poisson}(\lambda) \text{ with pmf } P(c \mid \lambda) = \frac{\lambda^c e^{-\lambda}}{c!}.$$

   The length of each call in hours, t_{ij}, is exponentially distributed with mean $1/\theta$:

$$t_{ij} \overset{\text{iid}}{\sim} \text{Exp}(\theta) \text{ with density } p(t \mid \theta) = \theta e^{-t\theta} \text{ for } j = 1, 2, \ldots, c_i.$$

   (a) (3 points) The prior for θ is Gamma(α, β): $p(\theta) = (\beta^\alpha / \Gamma(\alpha))\theta^{\alpha-1}e^{-\beta\theta}$. What is the posterior distribution for θ, given the call times on the first day (with c_1 known), $p(\theta \mid t_{11}, \ldots, t_{1c_1})$?
   This can be found using the Exponential-Gamma model, or derived directly:

$$\begin{aligned}
p(\theta \mid t_{11}, \ldots, t_{1c_1}) &\propto p(\theta) \prod_{j=1}^{c_1} p(t_{1j} \mid \theta) \\
&\propto \theta^{\alpha-1} e^{-\beta\theta} \theta^{c_i} e^{-(t_{11}+\cdots+t_{1c_1})\theta} \\
&= \theta^{\alpha+c_1-1} e^{-(\beta+t_{11}+\cdots+t_{1c_1})\theta} \\
&\propto \text{Gamma}\left(\alpha + c_1, \beta + \sum_{j=1}^{c_1} t_{1j}\right).
\end{aligned}$$

   (b) (2 points) After observing 5 calls on the first day, the *posterior* for λ is $p(\lambda \mid c_1 = 5) = \text{Gamma}(7, 2)$. What was the *prior* for λ, $p(\lambda)$?

   By the Poisson-Gamma model $p(\lambda \mid c_i) = \text{Gamma}(a + c_i, b + 1)$ if $p(\lambda) = \text{Gamma}(a, b)$, and so it follows that $p(\lambda) = \text{Gamma}(2, 1)$. This may also be derived analytically:

$$\begin{aligned}
\lambda^8 e^{-2\lambda} &\propto p(\lambda \mid c_1 = 5) \propto p(\lambda)p(c_1 = 5 \mid \lambda) \propto p(\lambda)\lambda^5 e^{-\lambda} \\
&\to p(\lambda) \propto \lambda^3 e^{-\lambda} \propto \text{Gamma}(2, 1).
\end{aligned}$$

   Let T_i denote the total call time on day i, $T_i = t_{i1} + \cdots + t_{ic_i}$. Let $\mathbf{y}_i$ denote all data from day i, $\mathbf{y}_i = \{c_i, t_{i1}, \ldots, t_{ic_i}\}$. For parts c-d below, assume that after the first day λ has posterior $p(\lambda \mid \mathbf{y}_1) = \text{Gamma}(7, 2)$ and θ has independent posterior $p(\theta \mid \mathbf{y}_1) = \text{Gamma}(4, 3)$.

(c) (3 points) What is the expected value for the length of a single call on day 2, given day 1, $E(t_{21} \mid \mathbf{y}_1)$?

$$\begin{aligned}
\int t_{21} p(t_{21} \mid y_1) dt_{21} &= \int t_{21} \int p(t_{21} \mid \theta) \cdot p(\theta \mid y_1) d\theta dt_{21} \\
&= \int \left[\int t_{21} p(t_{21} \mid \theta) dt_{21} \right] \cdot p(\theta \mid y_1) d\theta \\
&= \int (1/\theta) p(\theta \mid y_1) d\theta \\
&= \frac{3^4}{\Gamma(4)} \int \theta^2 e^{-3\theta} d\theta \\
&= \frac{3^4}{\Gamma(4)} \frac{\Gamma(3)}{3^3} = 1.
\end{aligned}$$

(d) (3 points) What is the expected value for the total call time on day 2, given day 1, $E(T_2 \mid \mathbf{y}_1)$? (You may use the result $E(T_2 \mid \mathbf{y}_1, c_2) = c_2$.)

$$\begin{aligned}
\int T_2 p(T_2 \mid y_1) dT_2 &= \int T_2 \left[\sum_{c_2} p(T_2 \mid c_2) p(c_2 \mid y_1) \right] dT2 \\
&= \sum_{c_2} \left[\int T_2 p(T_2 \mid c_2) dT_2 \right] P(c_2 \mid y_1) \\
&= \sum_{c_2} c_2 P(c_2 \mid y) \\
&= \sum_{c_2} c_2 \int P(c_2 \mid \lambda) p(\lambda \mid y_1) d\lambda \\
&= \int \left[\sum_{c_2} c_2 P(c_2 \mid \lambda) \right] p(\lambda \mid y_1) \\
&= \int \lambda p(\lambda \mid y_1) = 7/2.
\end{aligned}$$

2. *Bayesian P-Value* [3 pts]
Consider $y_1, \ldots, y_n \stackrel{iid}{\sim} N(\theta, \sigma^2)$, with σ^2 known and a flat prior on θ: $p(\theta) = 1 \ \forall \ \theta$. Let $\mathbf{y} = (y_1, \ldots, y_n)$, and let $T(\cdot)$ give the sample mean: $T(\mathbf{y}) = \bar{y}$. What is the posterior predictive p-value,

$$P(T(\mathbf{Y}) \geq T(\mathbf{y}) \mid \mathbf{y})?$$

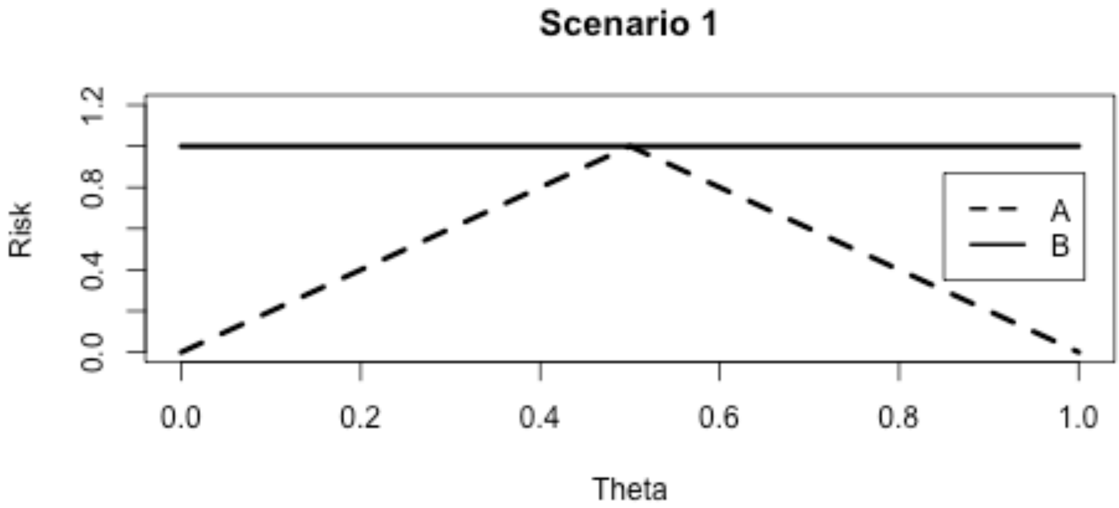
The posterior distribution of θ *given* $\mathbf{y}$ *is*

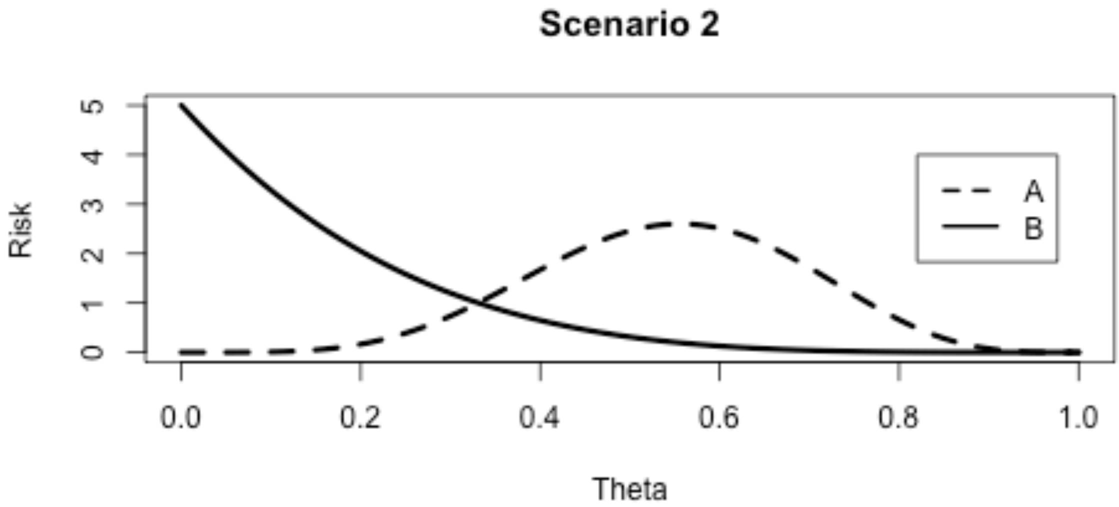
$$\theta \mid \mathbf{y} \sim N(\bar{y}, \sigma^2/n)$$

(see slide 9 of the "More on Interval Estimation" notes). So, because $\bar{Y} \sim N(\theta, \sigma^2/n)$, *it follows from the marginal distribution of the normal-normal model that*

$$\bar{Y} \mid \mathbf{y} \sim N(\bar{y}, 2\sigma^2/n).$$

Thus, $P(\bar{Y} \geq \bar{y} \mid \mathbf{y}) = \frac{1}{2}$.





3. *Decision rules* [6 pts]
The above figure shows the frequentist risk of two decision rules (A and B) as a function of an unknown parameter θ, under two different scenarios. For each scenario, A and B are the only two potential decisions (action space = {A,B}), θ can take any value between 0 and 1, and θ has a prior that is positive over the unit interval $p(\theta) > 0 \; \forall \; \theta \in [0, 1]$.

For each of the following, answer "yes", "no", or "not enough informations":

(a) Is A admissible in Scenario 1? *Yes*

(b) Is A minimax in Scenario 1? *Yes*

(c) Is A a Bayes rule in Scenario 1? *Yes*

(d) Is B admissible in Scenario 1? *No*

(e) Is B minimax in Scenario 1? *Yes*

(f) Is B a Bayes rule in Scenario 1? *No*

(g) Is A admissible in Scenario 2? *Yes*

(h) Is A minimax in Scenario 2? *Yes*

(i) Is A a Bayes rule in Scenario 2? *Not enough information*

(j) Is B admissible in Scenario 2? *Yes*

(k) Is B minimax in Scenario 2? *No*

(l) Is B a Bayes rule in Scenario 2? *Not enough information*