Rejection Sampling

PUBH 8442: Bayes Decision Theory and Data Analysis

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Background

- We would like to sample from a (potentially unknown) density p
- This is equivalent to sampling uniformly from the area under p
- If $p \propto q$, this is also equivalent to sampling uniformly from area under q
- In our context, p is a posterior distribution and q is the unnormalized version

$$q(\theta) = p(\mathbf{y} \mid \theta) p_\theta(\theta).$$

- How to sample uniformly underneath q?

Background

- Choose an *enveloping function* Mg that you can sample underneath

$$q(\theta) \leq Mg(\theta)$$

- Ignore those samples that are above q

- Those samples underneath q are uniform underneath q. Use these.

- In practice, g is a density function and M is a constant to assure Mg envelopes q

Rejection sampling

- Algorithm to simulate from posterior distribution $p(\theta \mid \mathbf{y})$:
  - Generate $\theta_j \sim g(\theta)$
  - Generate $U \sim \text{Uniform}(0, 1)$
    - Note $(\theta_j, UMg(\theta_j))$ is a random point under $Mg(\theta_j)$
  - If $MUg(\theta_j) < p(\mathbf{y} \mid \theta_j)p(\theta_j)$ accept θ_j; otherwise reject θ_j.
- The accepted θ_j correspond to draws from $p(\theta \mid \mathbf{y})$
- Repeat above process until desired number of (accepted) samples N is obtained

Rejection sampling

- Equivalent algorithm:

1. Generate $\theta_j \sim g(\theta)$
2. Accept with probability

$$p_j = \frac{p(\mathbf{y} \mid \theta_j)p(\theta_j)}{Mg(\theta_j)}$$

otherwise reject θ_j.

Comments

- Similar to importance sampling
  - Simulations from known density $g(\theta)$ used to approximate unknown $p(\theta \mid \mathbf{y})$
  - Importance sampling is analogous to using p_j as weights instead of acceptance probabilities
- Rejection sampling yields direct samples from $p(\theta \mid \mathbf{y})$ (not weighted samples)
- Envelope function $Mg(\theta)$ MUST satisfy $Mg(\theta) \geq p(\mathbf{y} \mid \theta)p(\theta)\ \forall\theta$
  - Otherwise results are inconsistent
- Ideally, M is as small as possible and $Mg(\theta)$ is close to $p(\theta \mid \mathbf{y})p(\theta)$

Simple example: normal density

- Goal: simulate samples from $p(\theta) = N(0, 1)$ using rejection sampling[1]

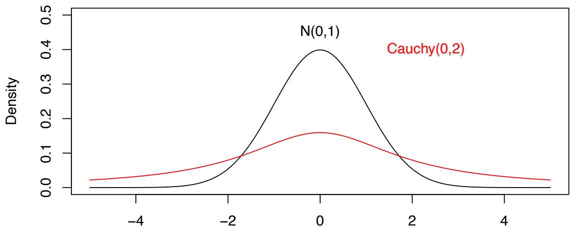
- Use $g(\theta) = \text{Cauchy}(0, 2)$ as envelope function.

$$g(\theta) = \frac{1}{2\pi \left[1 + \left(\frac{\theta}{2}\right)^2\right]}$$

[1] Example source: http://glau.ca/?p=227

Simple example: normal density

- Consider $M = 1$:

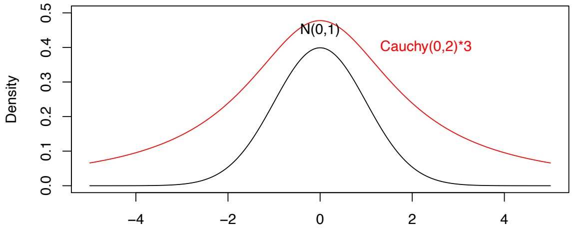


http://www.ericfrazerlock.com/Rejection_Sampling_Rcode1.r

- Not a proper envelope

Simple example: normal density

- Try $M = 3$:

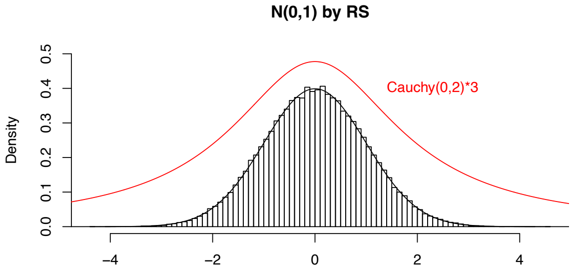


http://www.ericfrazerlock.com/Rejection_Sampling_Rcode1.r

- This is a proper envelope (though somewhat inefficient)

Simple example: normal density

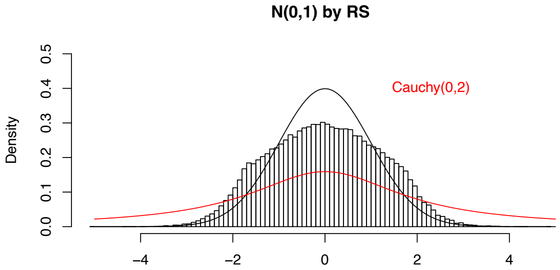
- Histogram of accepted samples using $M = 3$:



http://www.ericfrazerlock.com/Rejection_Sampling_Rcode1.r

Simple example: normal density

- Histogram of accepted samples using $M = 1$:



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