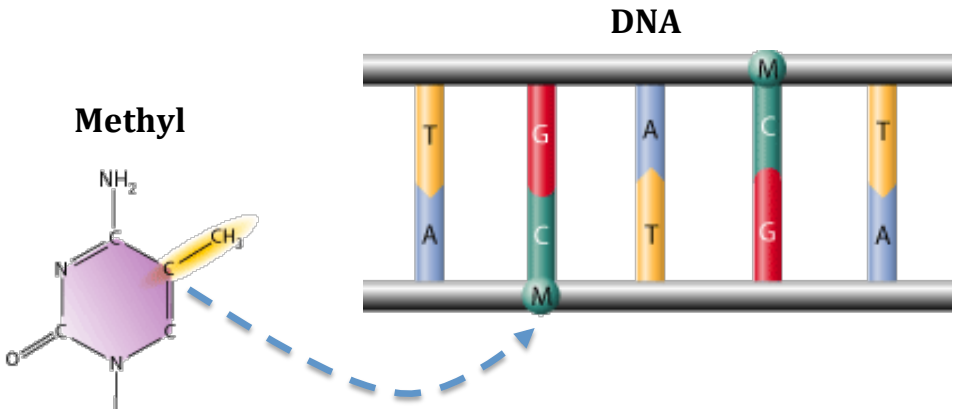
Bayesian Screening for Group Differences in High-Throughput Data

April 27, 2025

DNA Methylation

- Methyl binds to CpG (cytosine-phosphate-guanine) sites



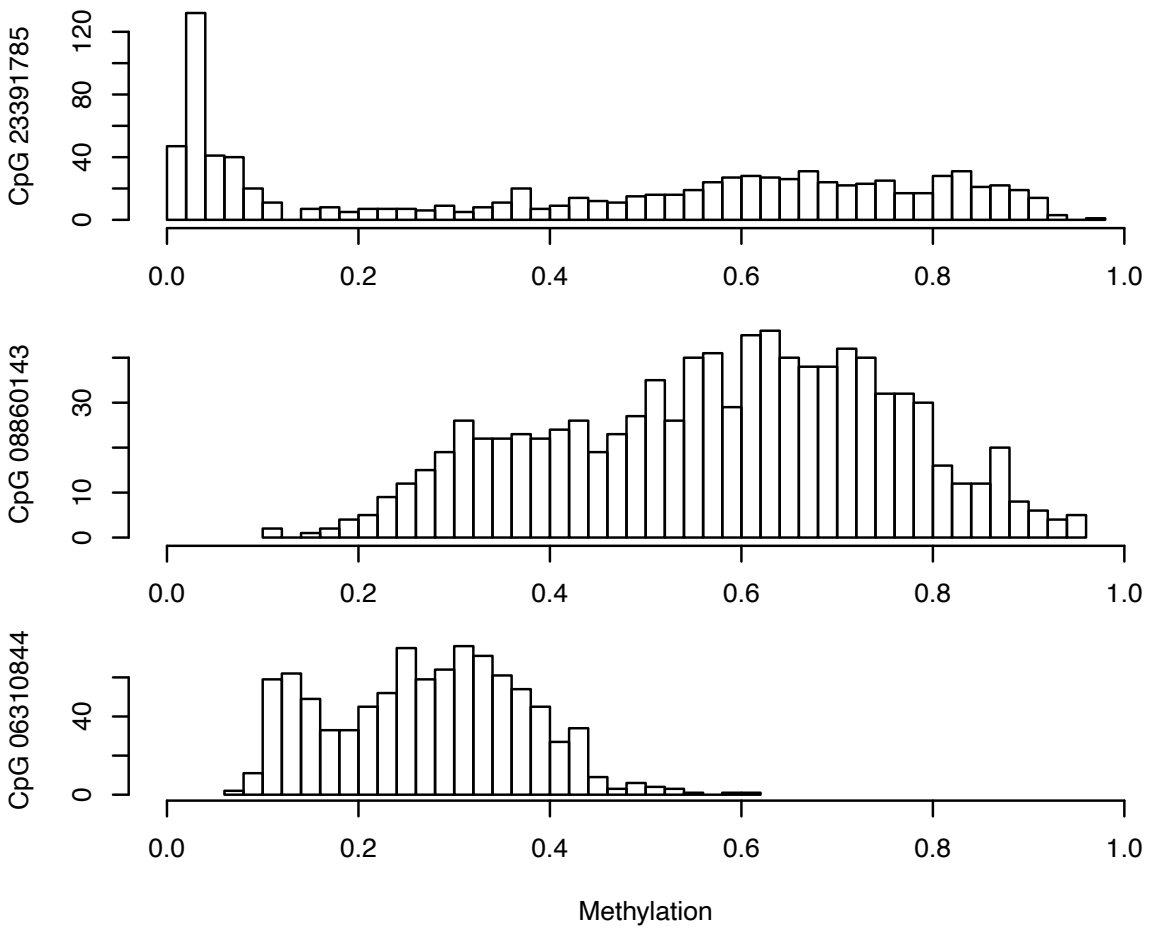
- Over 25 million CpG sites in human genome
- Methylation varies over sites / individuals / cell types
- Can affect gene transcription

TCGA array data: BRCA

- $N = 597$ breast cancer tumor samples
  - From The Cancer Genome Atlas project
- Methylation measured for $M = 21,986$ CpG sites
  - Illumina HumanMethylation27 array
  - Measurements from 0 (no methylation) to 1 (fully methylated)
- Goal: study role of methylation in clinical heterogeneity
  - Basal ($N_0 = 112$) vs. non-Basal ($N_1 = 485$) tumor subtypes

Example distributions

- Distribution of methylation values for select CpG sites



Kernel mixtures

- Model distribution of CpG m ($m = 1, \ldots, M$) as a mixture:

$$x_{mn} \sim \sum_{k=1}^{K} \pi_{mk} F_k$$

  - $\{F_k\}_{k=1}^{K}$ are shared kernels
  - $\Pi_m = \{\pi_{mk}\}_{k=1}^{K}$ are CpG-specific weights

- F_k is Normal(μ_k, σ_k) truncated between 0 and 1

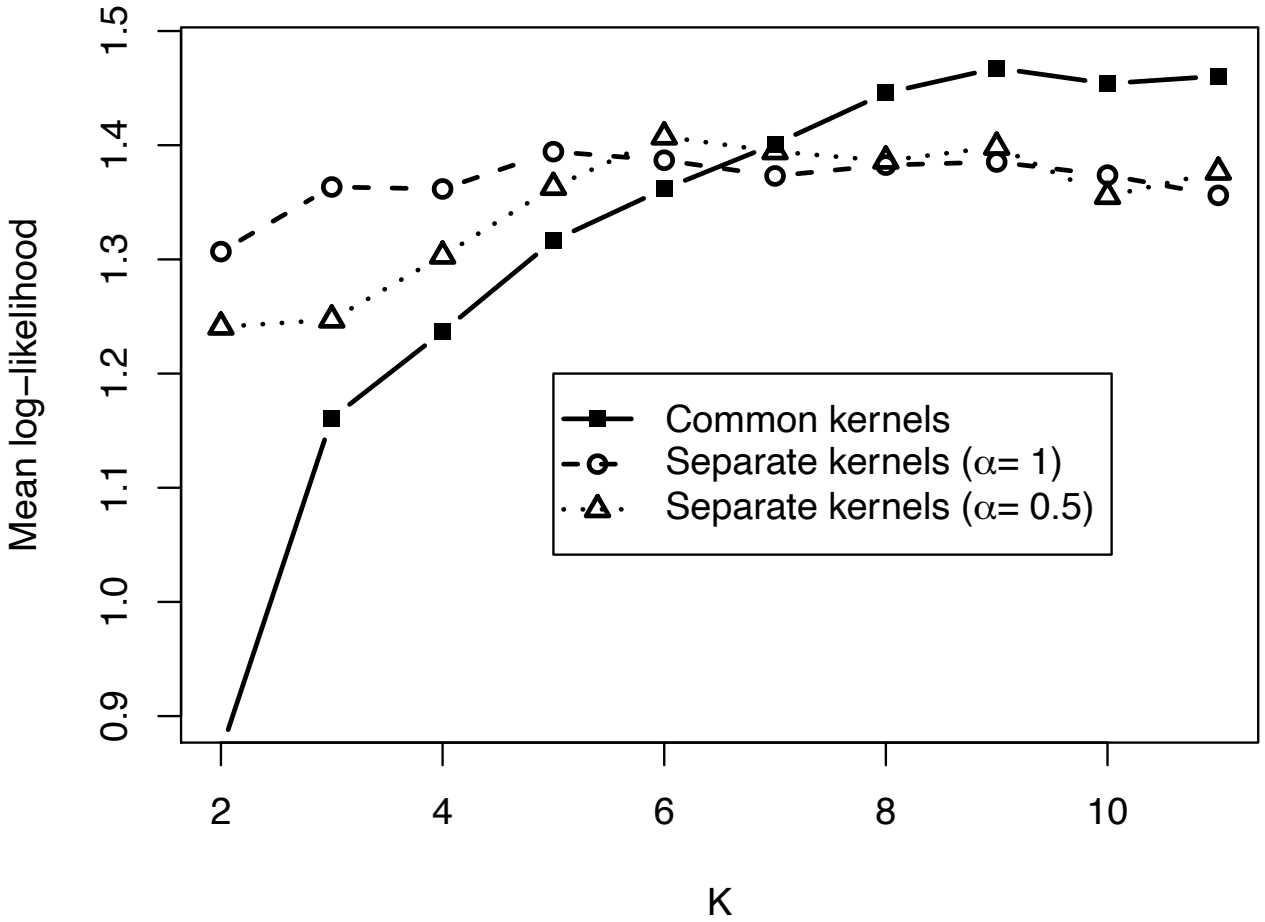
Bayesian estimation

- Use normal-inverse-gamma prior for (μ_k, σ_k)'s
- Use Dirichlet(α) prior for Π_m's
- Gibbs sample from conditional posteriors of
  - $\{(\mu_k, \sigma_k)\}_{k=1}^{K}$
  - $\{\Pi_m\}_{m=1}^{M}$
  - Kernel memberships $\{C_m\}_{m=1}^{M}$
- Estimate α via maximum likelihood during sampling

Choice of K

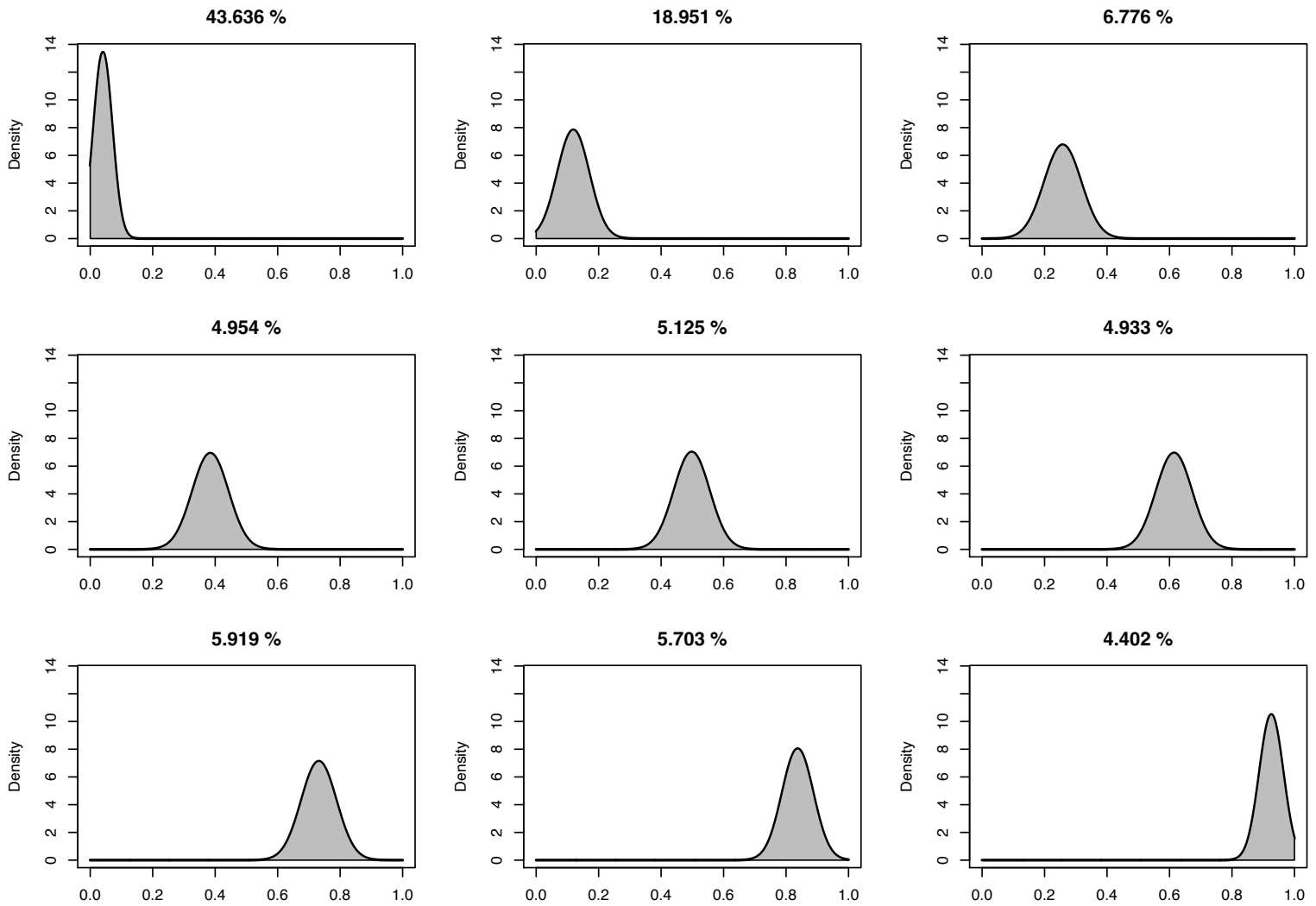
- Choose K to maximize likelihood under cross validation.
- For fixed K:
  - Estimate $F_1, \ldots, F_K$, and α from a sub-sample of CpGs
  - For each remaining CpG:
    - Hold out a random observation
    - Estimate kernel weights on $N - 1$ remaining observations
    - Compute log-density for held out sample
  - Consider mean log-density for all held-out observations

Cross-validated log-likelihood

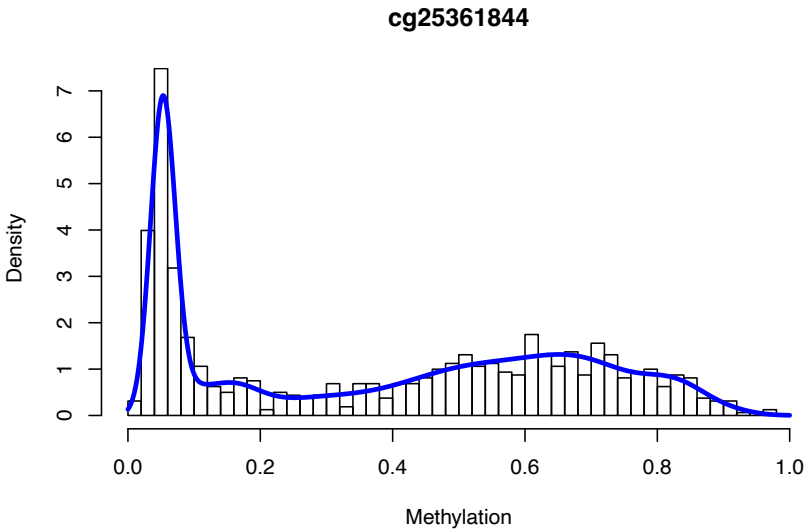


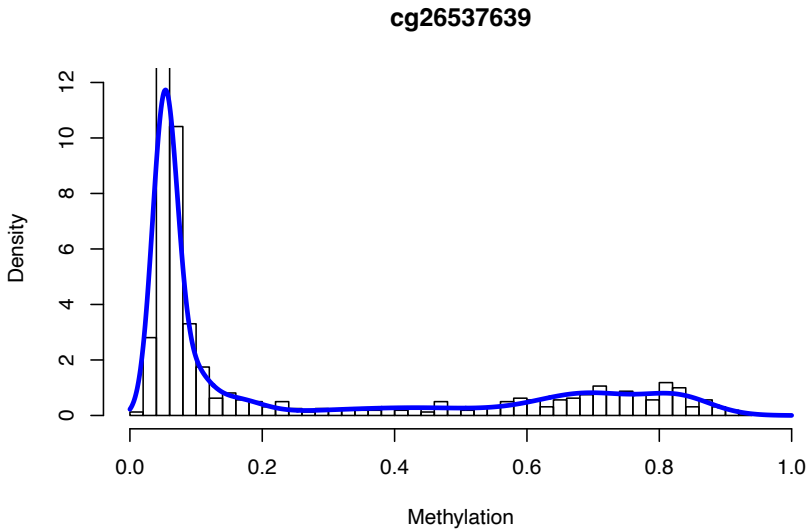
- Choose $K = 9$

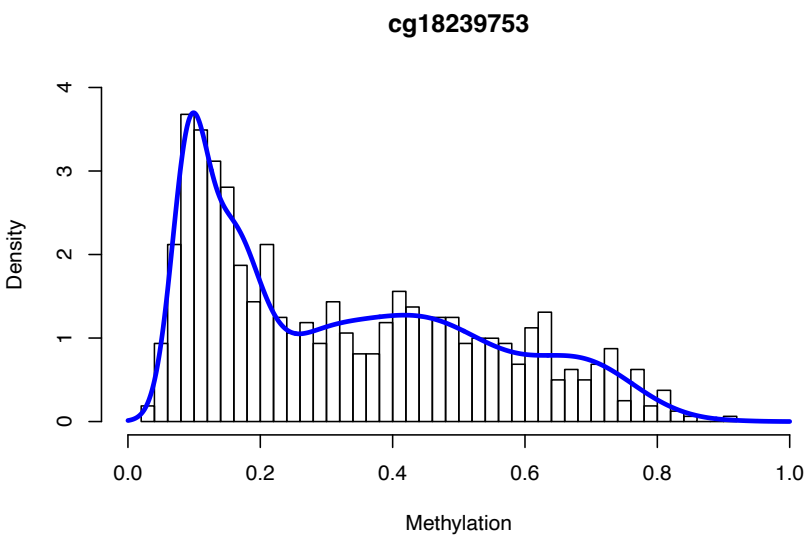
Kernel distributions

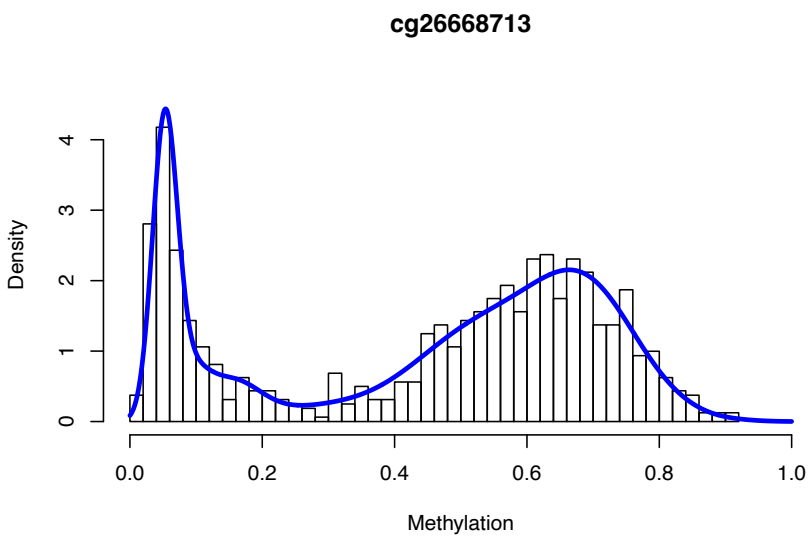


Fitted mixture examples



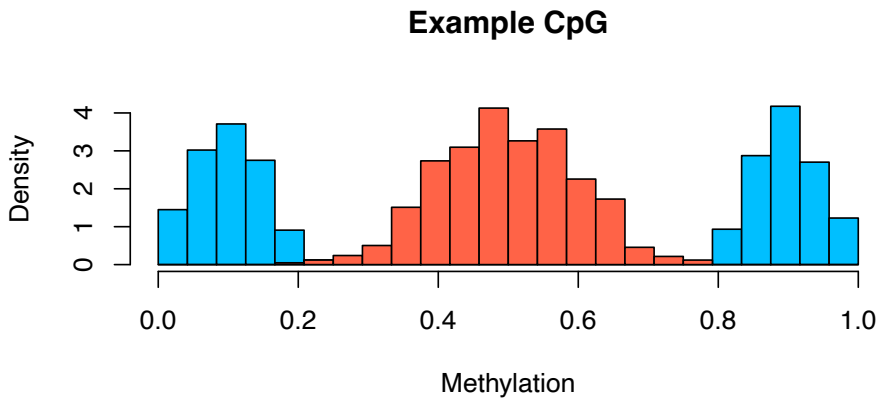






Test for group equality

- For group comparisons at a CpG, t- and Wilcoxon tests are most common
  - Bock 2012, Laird 2013
- General tests for distributional equality are rarely used
- But they are well motivated...
  - Cancer & normal cells show different variability (Hansen 2011)
  - Groups may have differential "stability" across cells:



Test for group equality

- Compare Basal vs. non-Basal tumor subtypes at each CpG
  - Assess whether subtype distributions are different
- Subtype distributions $F_m^{(0)}, F_m^{(1)}$ are mixture of common kernels

$$F_m^{(0)} = \sum_{k=1}^{K} \pi_{mk}^{(0)} F_k \text{ and } F_m^{(1)} = \sum_{k=1}^{K} \pi_{mk}^{(1)} F_k,$$

- For each m test

$$H_{0m} : \pi_{mk}^{(0)} = \pi_{mk}^{(1)} \text{ for all } k$$
$$H_{1m} : \pi_{mk}^{(0)} \neq \pi_{mk}^{(1)} \text{ for some } k.$$

Bayesian framework

- Estimate and fix $F_1, \ldots, F_K$, and α as before.
- Under H_{0m}, $\Pi_m^{(0)} = \Pi_m^{(1)} = \Pi_m \sim \text{Dirichlet}(\alpha)$
- Under H_{1m}, $\Pi_m^{(0)}, \Pi_m^{(1)} \sim \text{Dirichlet}(\alpha)$ are independent
- P_0 is shared prior probability of equality at a given CpG
  - P_0 given Uniform$(0, 1)$ prior (see Scott & Berger 2010)

Posterior computation

- The full conditional posterior probability for H_{0m} is

$$\frac{P_0\beta(\alpha)\beta(\vec{n}_m+\alpha)}{P_0\beta(\alpha)\beta(\vec{n}_m+\alpha)+(1-P_0)\beta(\vec{n}_m^{(0)}+\alpha)\beta(\vec{n}_m^{(1)}+\alpha)}.$$

  - $\vec{n}_m^{(i)}$ gives number of realizations in group i from each kernel
  - $\vec{n}_m = \vec{n}_m^{(0)} + \vec{n}_m^{(1)}$
  - β is the multivariate beta function

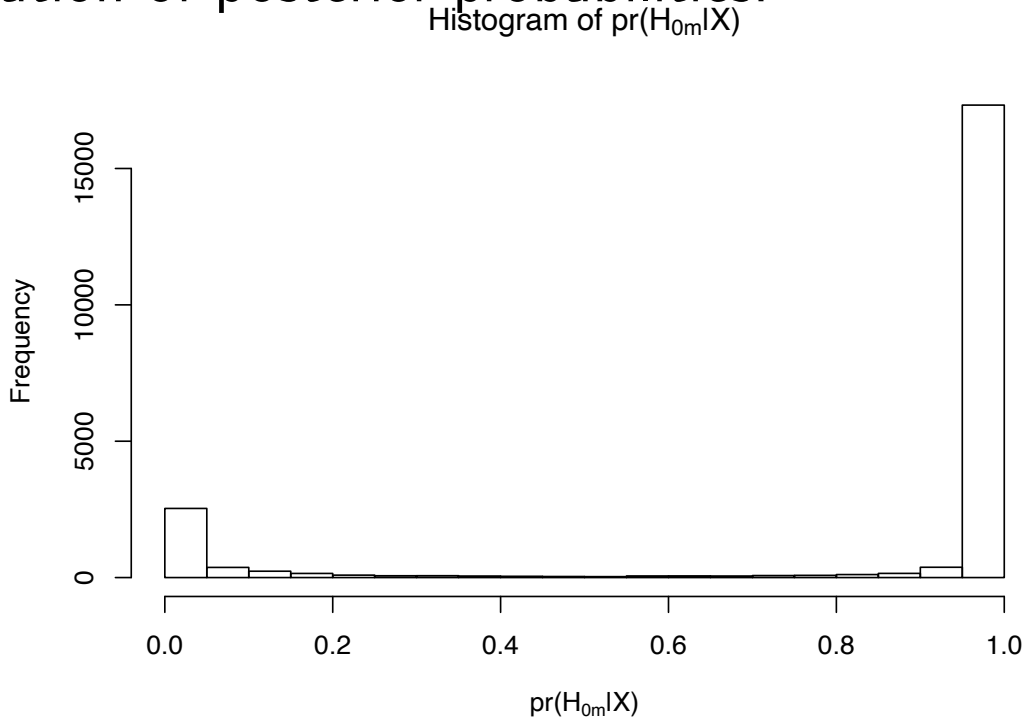
$$\beta(\alpha) = \frac{\prod_{k=1}^{K}\Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{K}\alpha_k)}.$$

Posterior computation

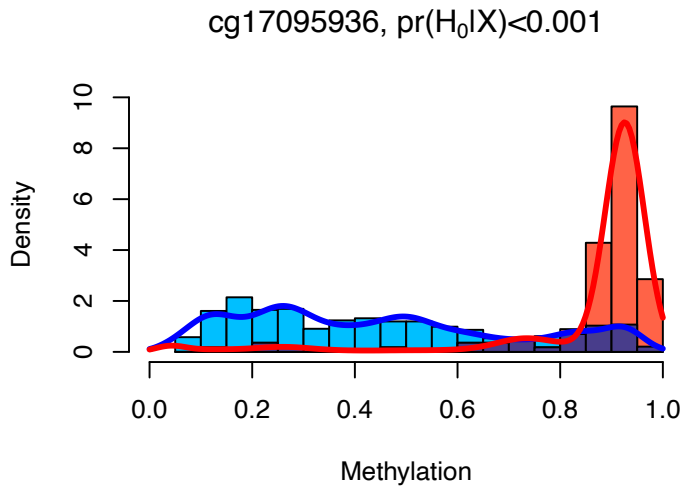
- In practice $\vec{n}_m^{(0)}$, $\vec{n}_m^{(1)}$ are unknown
- Kernel memberships are inferred probabilistically
- Gibbs sample from conditional posteriors of
  - $\{\Pi_m^{(0)}, \Pi_m^{(1)}\}_{m=1}^M$
  - $\{\vec{n}_m^{(0)}, \vec{n}_m^{(1)}\}_{m=1}^M$
  - $\{P(H_{0m} \mid \vec{n}_m^{(0)}, \vec{n}_m^{(1)})\}_{m=1}^M$
  - P_0
- Average over conditional posterior probabilities for H_{0m}

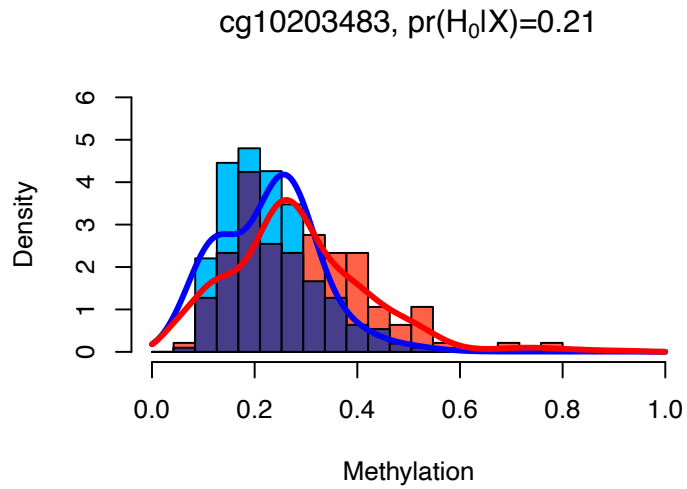
Basal vs. non-Basal groups

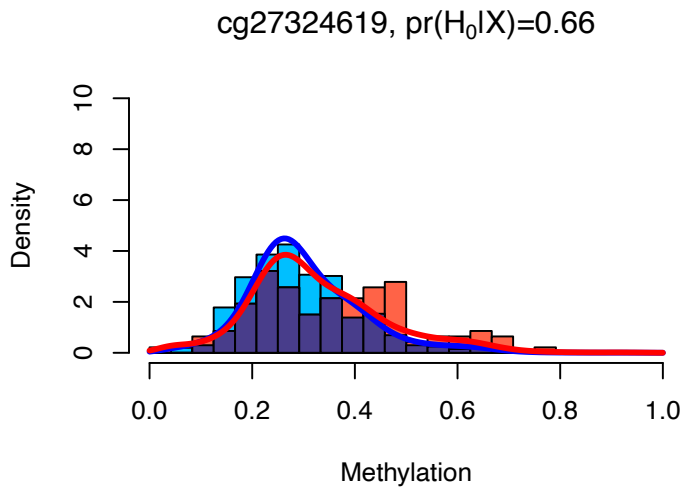
- Prior probability of equality: $\hat{P}_0 = 0.82$
- Distribution of posterior probabilities:

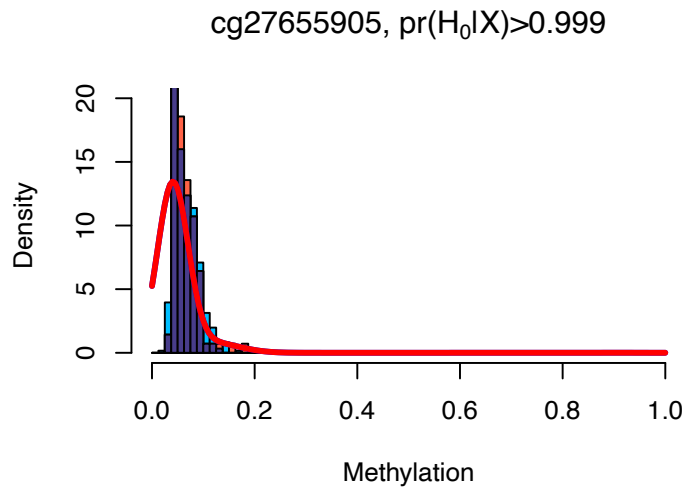


Basal vs. non-Basal groups



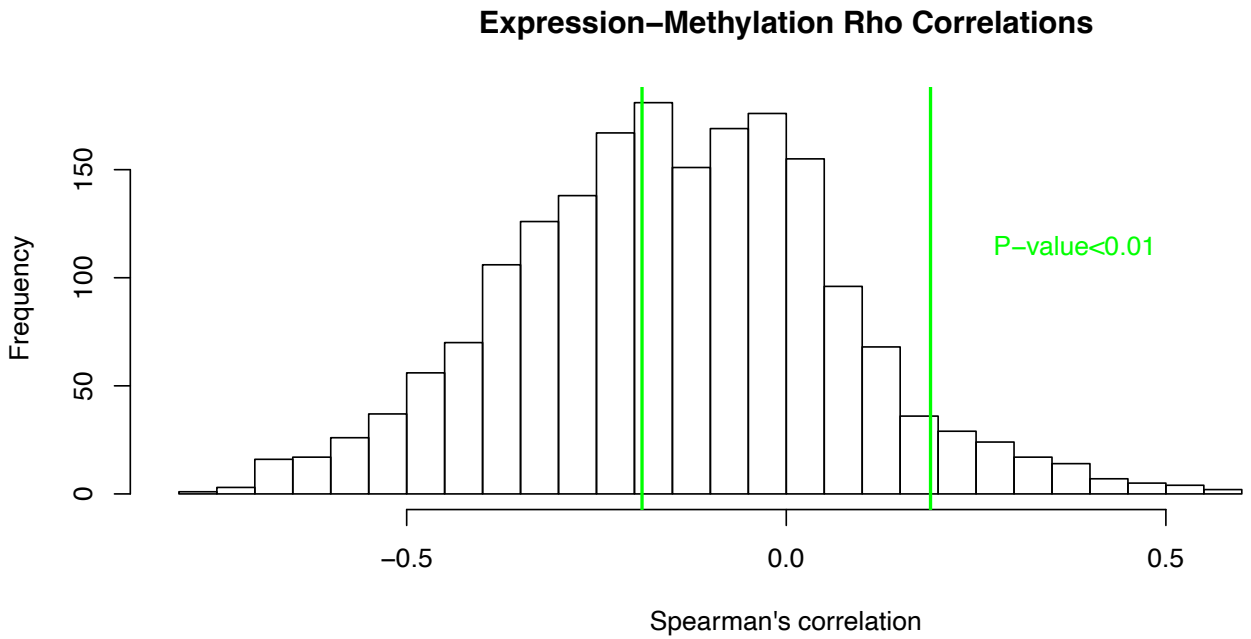






Basal vs. non-Basal groups

- 2117 CpG sites with $P(H_{0m}|X) < 0.01$
- Consider association with expression at their gene:



- Negative association & in PAM50 signature (Parker, 2009):
  - *MYBL2*, *EGFR*, *MIA*, *SFRP1* and *MLPH*

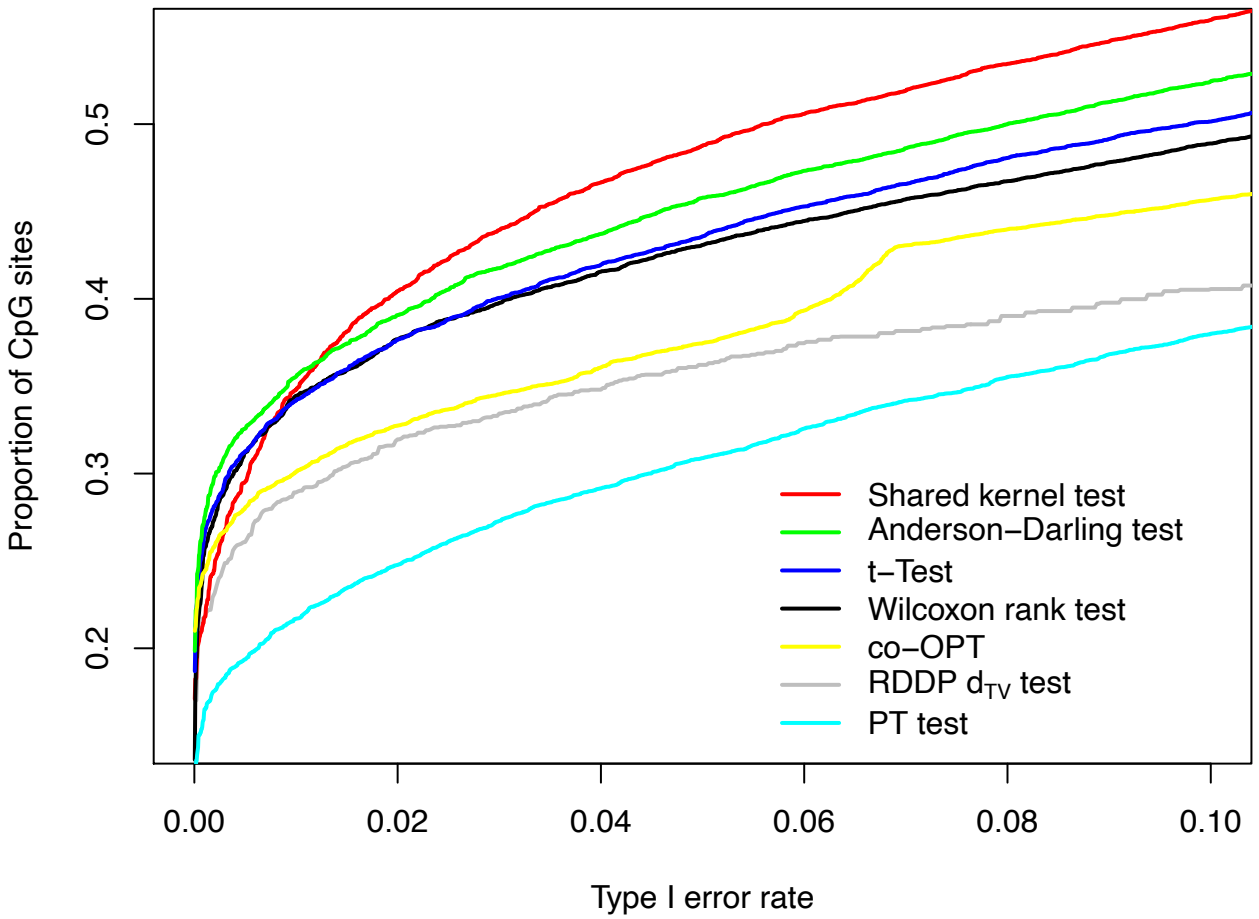
Related work

- Frequentist tests for distributional equality
  - Anderson-Darling, Shapiro-Wilk
- Bayesian nonparametric tests using Dirichlet processes
  - Dunson & Peddada 2008, Pennell & Dunson 2008
- Bayesian nonparametric tests using Polya trees
  - Ma & Wang 2011, Holmes et al 2014

Methods comparison for TCGA data

- Apply several methods to TCGA data
  - t-test, Wilcoxon test, Anderson-Darling test, Dunson & Peddada (RDDP), Ma & Wang (co-OPT), Holmes et al. (PT), and shared kernel test with fixed $P_0 = 0.5$.
- Permute class labels for each CpG and apply again.
- Permutation creates a null model to assess type I error
- Compare distribution of results (p-values or Bayes factors) for true and permuted data.

Methods comparison for TCGA data



THEORETICAL INTERLUDE*

Abstract testing framework

- Two distributions $F^{(0)}, F^{(1)}$ are mixtures

$$F^{(0)} = \sum_{k=1}^{K} \pi_k^{(0)} F_k \text{ and } F^{(1)} = \sum_{k=1}^{K} \pi_k^{(1)} F_k,$$

- Test whether $\pi_k^{(0)} = \pi_k^{(1)} \ \forall$ k.
- $F^{(0)}, F^{(1)}$ describe two populations with same strata
  - Test whether strata have different proportions

Abstract testing framework

- If strata/kernel memberships are known:
  - Test for association in $2 \times K$ table
  - Frequentist approaches: Chi-Square, Fisher's exact test
  - Bayesian Approaches: Good & Crook 1987, Albert 1997
- If memberships (and perhaps the F_k's) are unknown:
  - Little statistical literature
  - Addressed partly in Xu et al 2010

Asymptotic forms

- Consider behavior of the full conditional for H_0:

$$\frac{P_0\beta(\alpha)\beta(\vec{n}+\alpha)}{P_0\beta(\alpha)\beta(\vec{n}_m+\alpha)+(1-P_0)\beta(\vec{n}^{(0)}+\alpha)\beta(\vec{n}^{(1)}+\alpha)}$$

  as $N \to \infty$.

- For the following assume:
  - $\lambda_0 = \frac{N_0}{N_0+N_1}$ is fixed as $N_0 + N_1 = N \to \infty$

Asymptotic forms

- THEOREM: Can derive a closed asymptotic form for the full conditional
- CORROLARY: Can fully characterize asymptotic distribution under H_0 and H_1
- Under $H_0 : \Pi^{(0)} = \Pi^{(1)} = \Pi$, the log Bayes factor has order

$$\frac{K-1}{2}\log(N) + O_p(1)$$

- Under $H_1 : \Pi^{(0)} \neq \Pi^{(1)}$, let $\Pi^* = \lambda_0 \Pi^{(0)} + (1-\lambda_0)\Pi^{(1)}$.
  The log of the Bayes factor has order

$$-N \sum \left\{ \lambda_0 \pi_k^{(0)} \log\left(\frac{\pi_k^{(0)}}{\pi_k^*}\right) + (1-\lambda_0)\pi_k^{(1)} \log\left(\frac{\pi_k^{(1)}}{\pi_k^*}\right) \right\} + O_p\left(N^{1/2}\right),$$

Asymptotic forms

- Posterior probability of H_0 converges
  - Sublinearly to 1 under H_0
  - Exponentially to 0 under H_1
- Such rates have been observed for several Bayesian tests
  - Kass & Raftery 1995; Walker 2004; Johnson & Rossell 2010.
- Often such models are "local prior densities"
  - The parameter space under H_0 has positive density under H_1

Consistency under misspecification

- Bayesian context:
  - True distribution is not within support of prior
- E.g: data may not result from a finite Gaussian mixture
- Misspecified models not "fully" consistent
- May still be consistent as a test for distributional equality

Consistency under misspecification

- Use work of Kleijn & Van der Vaaart (2006)
- General behavior under Bayesian misspecification:
  - Let $\mathbb{F}$ be space of all distributions admitted by prior
  - Let F_0 be data generating distribution
  - Let F^* be distribution in $\mathbb{F}$ minimizing KL-divergence to F_0
  - Posterior concentrates on F^* as $N \to \infty$
- Little work on misspecification asymptotics for Bayesian tests

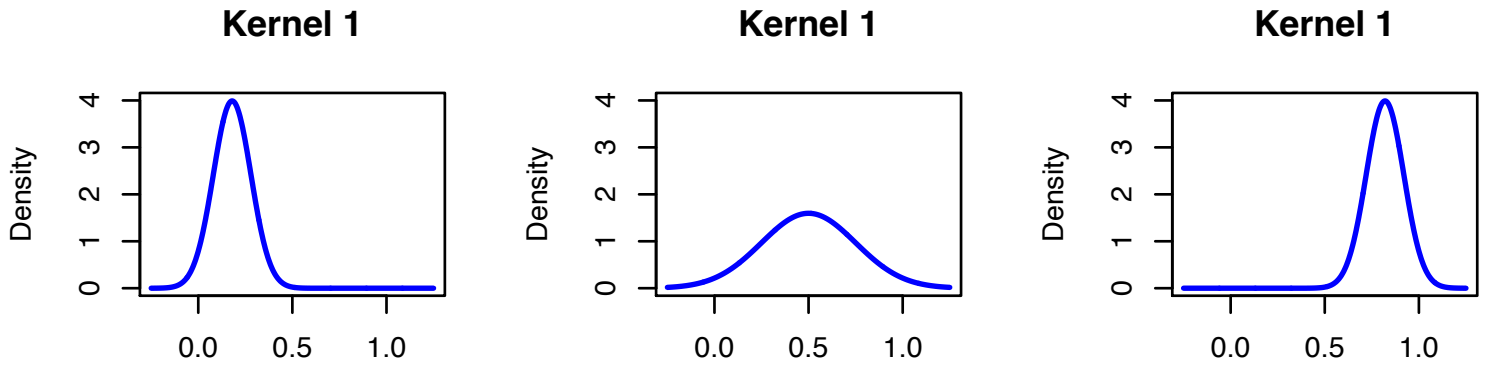
Misspecification for finite mixtures

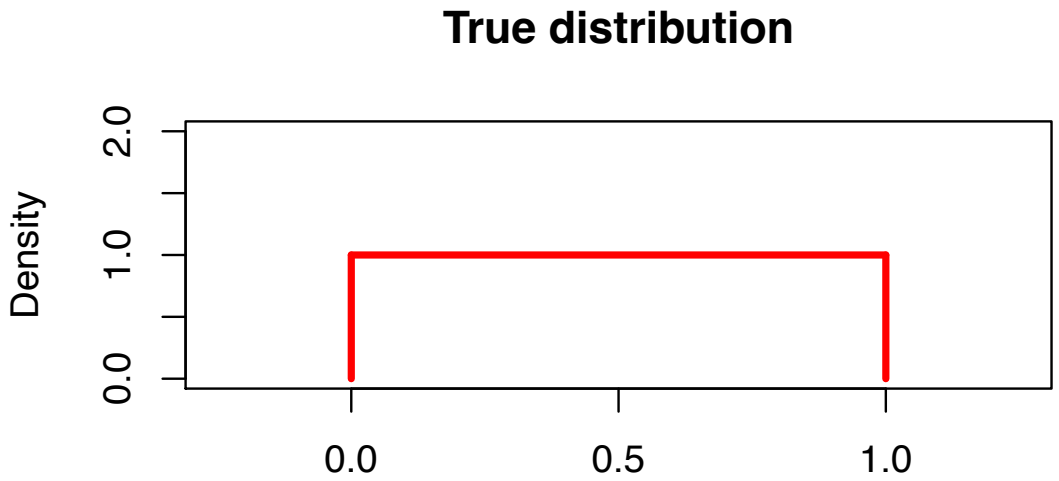
- Let $x_1, \ldots, x_N$ be independent with density f_0.
- Let $\mathbb{F}$ be define all convex combinations of densities $\{f_k\}_{k=1}^K$
- Let P define a prior with positive support over $\mathbb{F}$.
- Let $f^* = \underset{f \in \mathbb{F}}{\text{argmin}}\ \text{KL}(f_0 || f^*)$
- THEOREM: let $\Pi^* = (\pi_1^*, \ldots, \pi_K^*)$ be the component weights corresponding to f^*. Assume Π^* is unique in that $\sum \pi_k f_k = \sum \pi_k^* f_k = f^*$ only if $\Pi = \Pi^*$. Then, for any fixed $\epsilon > 0$,

$$\text{pr}(\Pi \in \mathbb{S}^{K-1} : ||\Pi - \Pi^*|| \geq \epsilon \mid x_1, \ldots, x_N) \to 0.$$

- Π^* is generally unique for normal $f_k's$ (Yakowitz 1968)

Illustrative example





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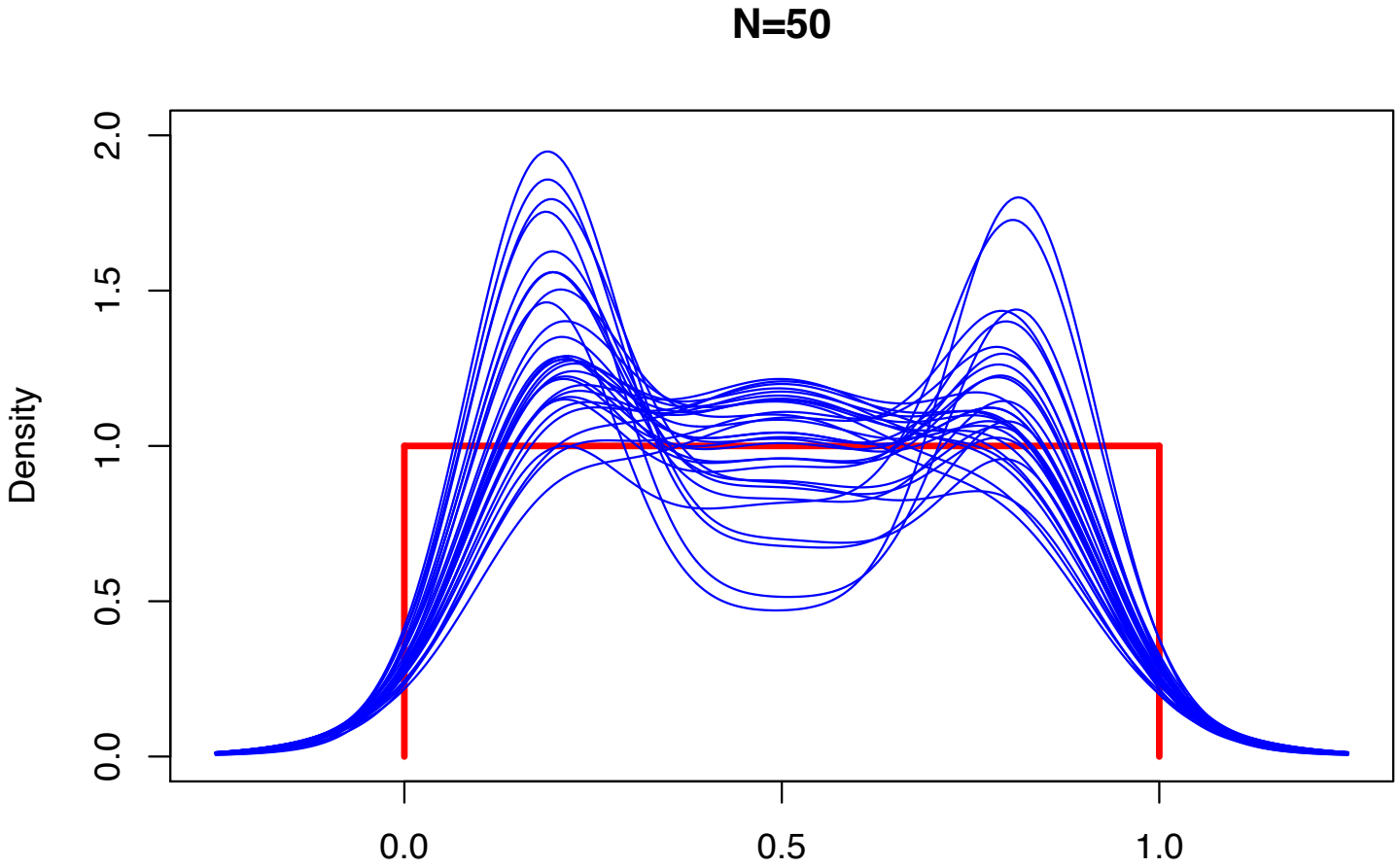

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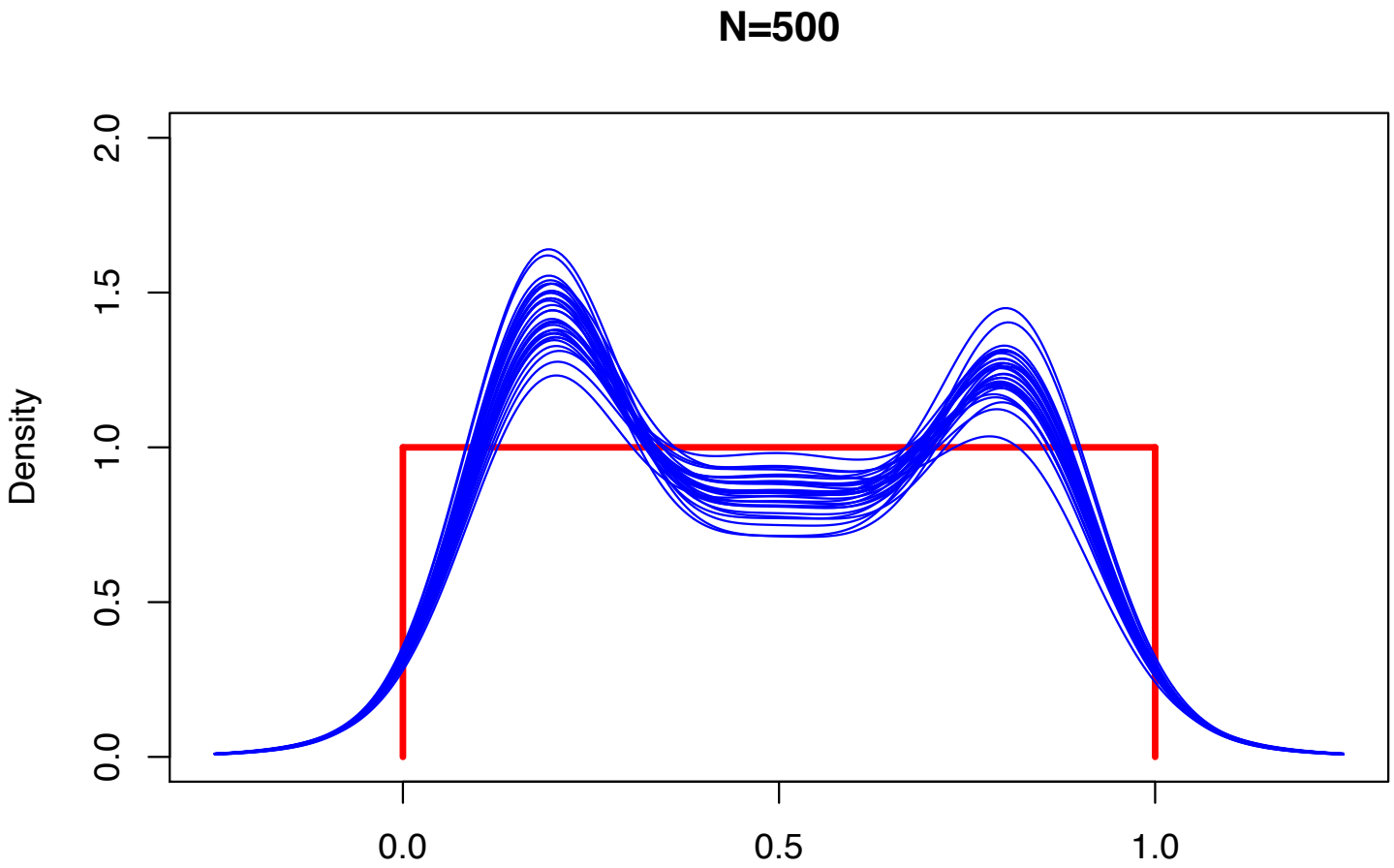

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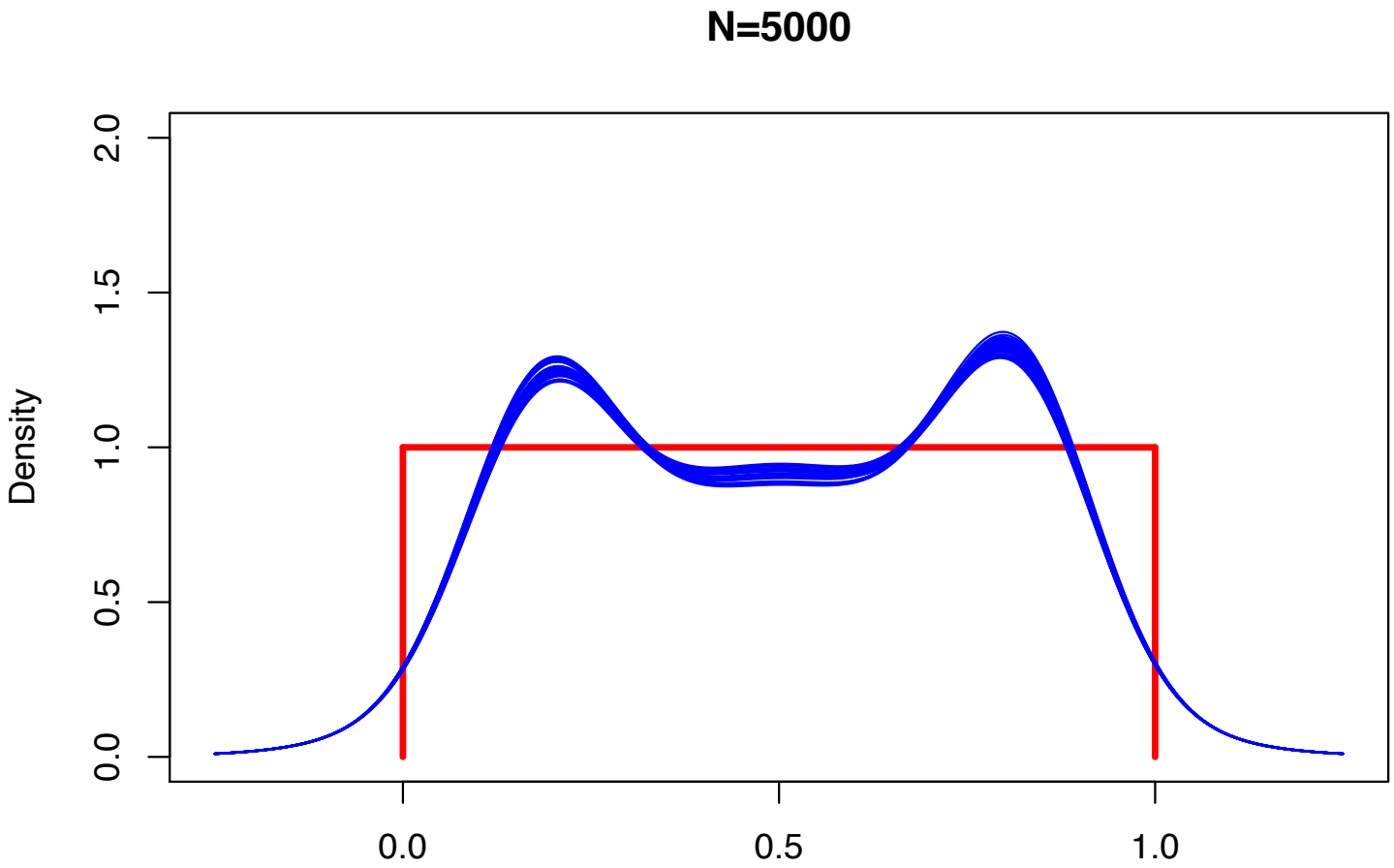


Misspecification for finite mixtures

- REMARK: Assume $\pi_k^* > 0$ for $k = 1, \ldots, K$ and $\sum \pi_k^* = 1$. Then, $f^* = \sum \pi_k^* f_k$ achieves the minimum KL-divergence in $\mathbb{F}$ with respect to f_0 if and only if

$$\int \frac{f_1}{f^*} f_0 = \ldots = \int \frac{f_K}{f^*} f_0.$$

  If some $\pi_k^* = 0$, the minimum KL-divergence is achieved where $\int \frac{f_k}{f^*} f_0$ are equivalent for all $\pi_k^* > 0$.

Consistency under misspecification

- THEOREM: Assume $x_1^{(0)}, \ldots, x_{N_0}^{(0)}$ are independent with density $f^{(0)}$, $x_1^{(1)}, \ldots, x_{N_1}^{(1)}$ are independent with density $f^{(1)}$, and let

$$f^{*(0)} = \underset{f \in \mathbb{F}}{\text{argmin}}\ \text{KL}(f^{(0)} || f)\ ,\ \ f^{*(1)} = \underset{f \in \mathbb{F}}{\text{argmin}}\ \text{KL}(f^{(1)} || f).$$

  Under uniqueness assumptions for $f^{*(0)}$ and $f^{*(1)}$,

  - if $f^{(0)} = f^{(1)}$, $\text{pr}(H_0 \mid X) \to 1$ as $N \to \infty$ and
  - if $f^{*(0)} \neq f^{*(1)}$, $\text{pr}(H_0 \mid X) \to 0$ as $N \to \infty$.

END THEORETICAL INTERLUDE

TCGA array data: Glioma

- $N = 258$ glioma tumor samples derived from astrocyte cells
- Methylation measured for $M \approx 450,000$ CpG sites
  - Illumina HumanMethylation450 array
  - Map to $\approx$ 20000 different genes
  - Sites per gene ranges from 1 to 1032
- Goal: study role of methylation in clinical heterogeneity
  - Lower grade gliomas (LGG) ($N_0 = 128$) vs. Glioblastoma Multiforme (GBM) ($N_1 = 130$) tumors

Hierarchical prior for distributional equality

- Model shared prior probability for all $450,000$ CpGs?

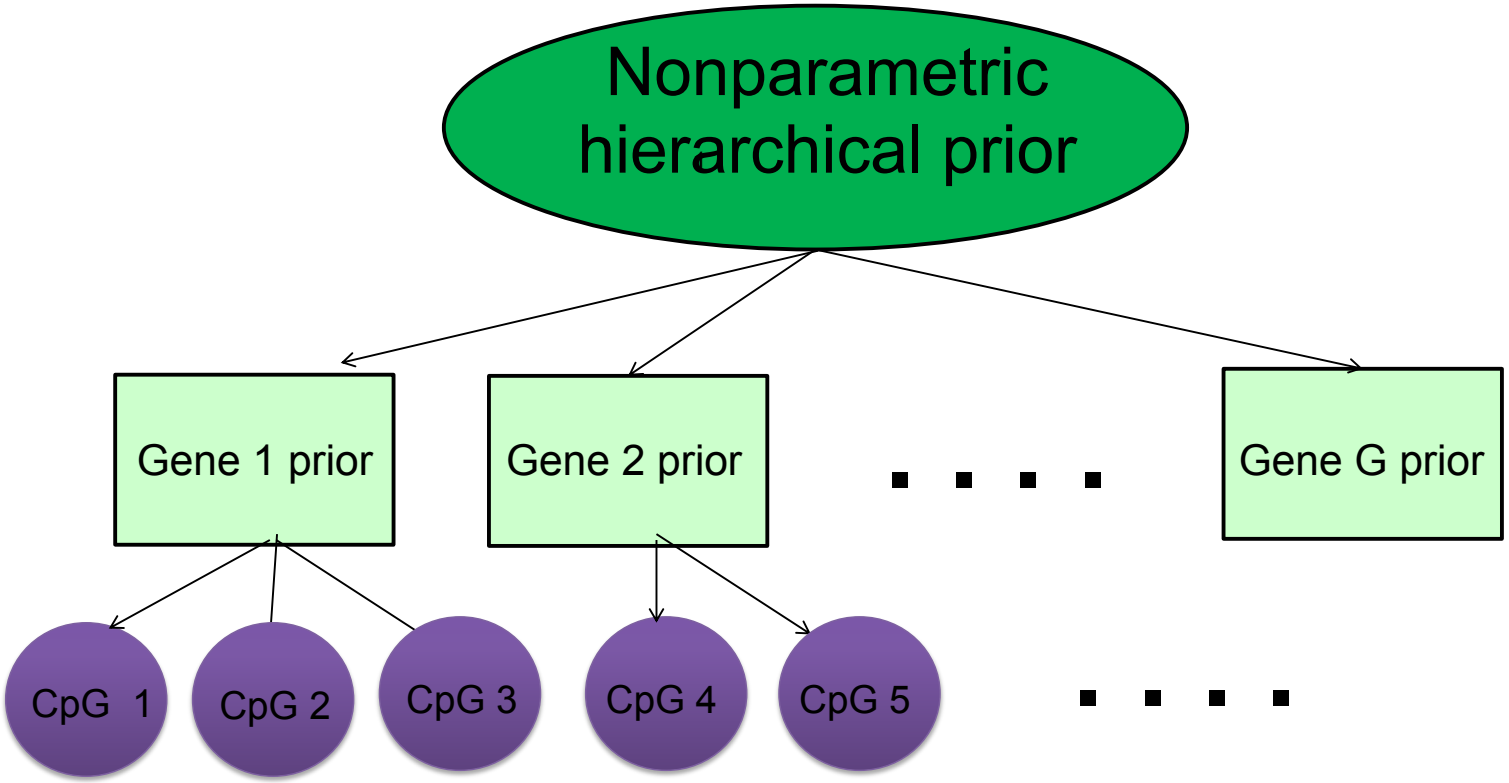
$$P_0 \sim \text{Beta}(1, 1)$$

- ...or separate prior probabilities for each gene?

$$P_{0g} \overset{iid}{\sim} \text{Beta}(1, 1) \quad \text{for } g = 1, \ldots, G$$

Hierarchical prior for distributional equality

- Hierarchical compromise:



Hierarchical prior for distributional equality

- Dirichlet process (DP) prior with Beta base distribution:

$$p_g \overset{iid}{\sim} P,$$
$$P \sim \text{DP}(\text{Beta}(a, b), \alpha)$$

- Equivalently,

$$p_g = \sum_{h=1}^{\infty} \pi_h \delta_{\theta_h},$$

  - δ_{θ_h} is a point mass at θ_h
  - $\theta_h \overset{iid}{\sim} \text{Beta}(a, b)$
  - Weights π_h realized from a *stick-breaking process*:

$$\pi_h = V_h \prod_{l<h} (1 - V_l)$$
$$V_h \overset{iid}{\sim} \text{Beta}(1, \alpha).$$

DP prior: hyperparameters

- Beta(a, b) base controls marginal prior of association

$$P(\text{CpG association}) = \frac{a}{a+b}.$$

- Concentration α controls level of clustering
  - $\alpha \to 0$: shared Beta(a, b) prior for all markers

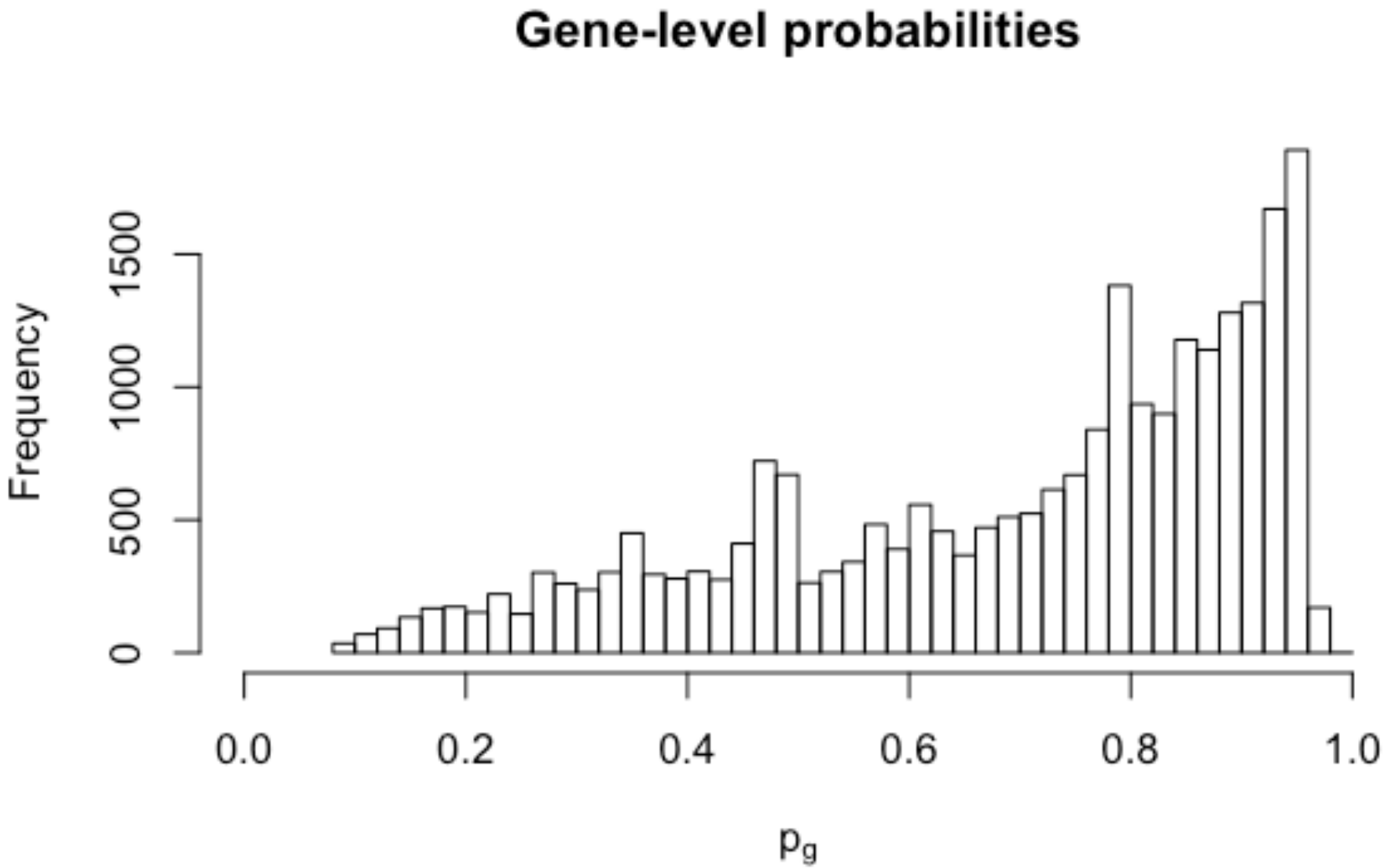
$$p_1 = \cdots = p_G \sim \text{Beta}(a, b)$$

  - $\alpha \to \infty$: independent Beta(a, b) prior for each gene

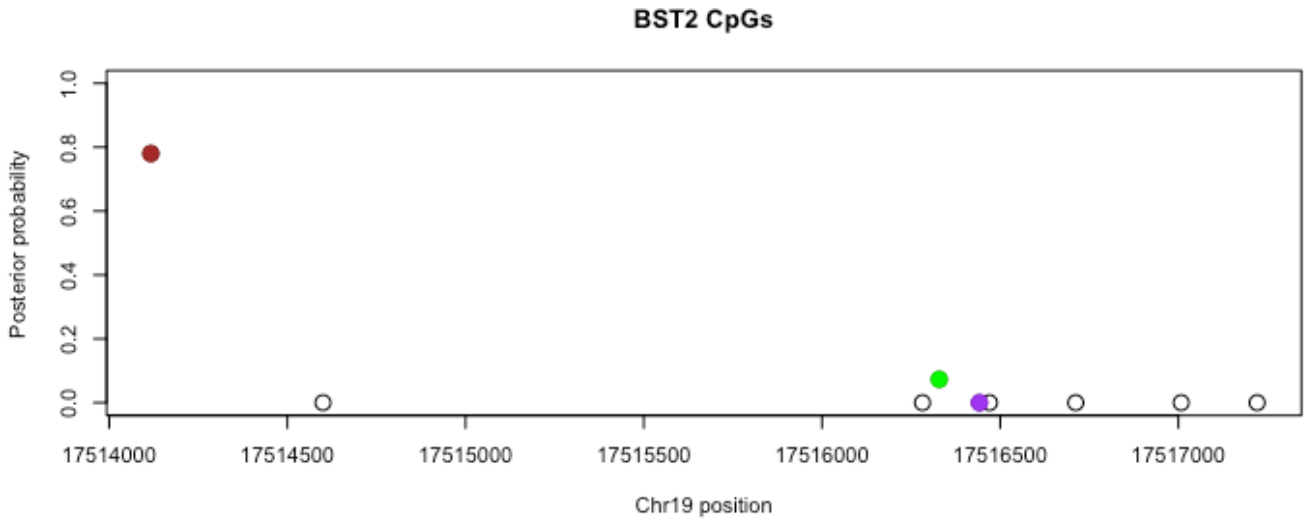
$$p_g \overset{iid}{\sim} \text{Beta}(a, b)$$

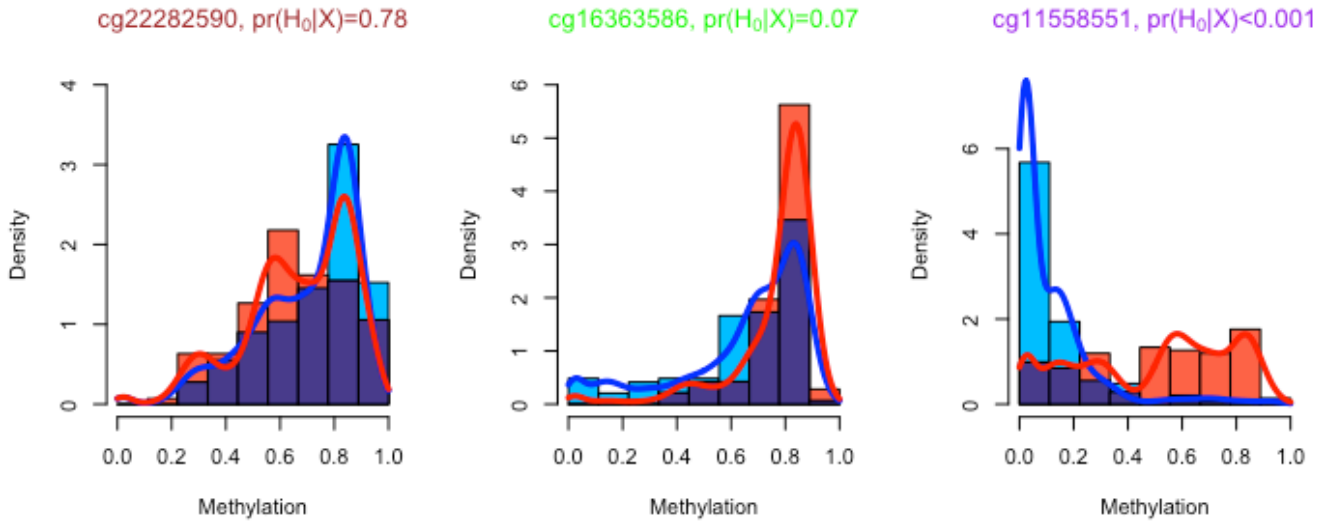
- In practice set $a = b = \alpha = 1$

TCGA Glioma analysis



TCGA Glioma analysis

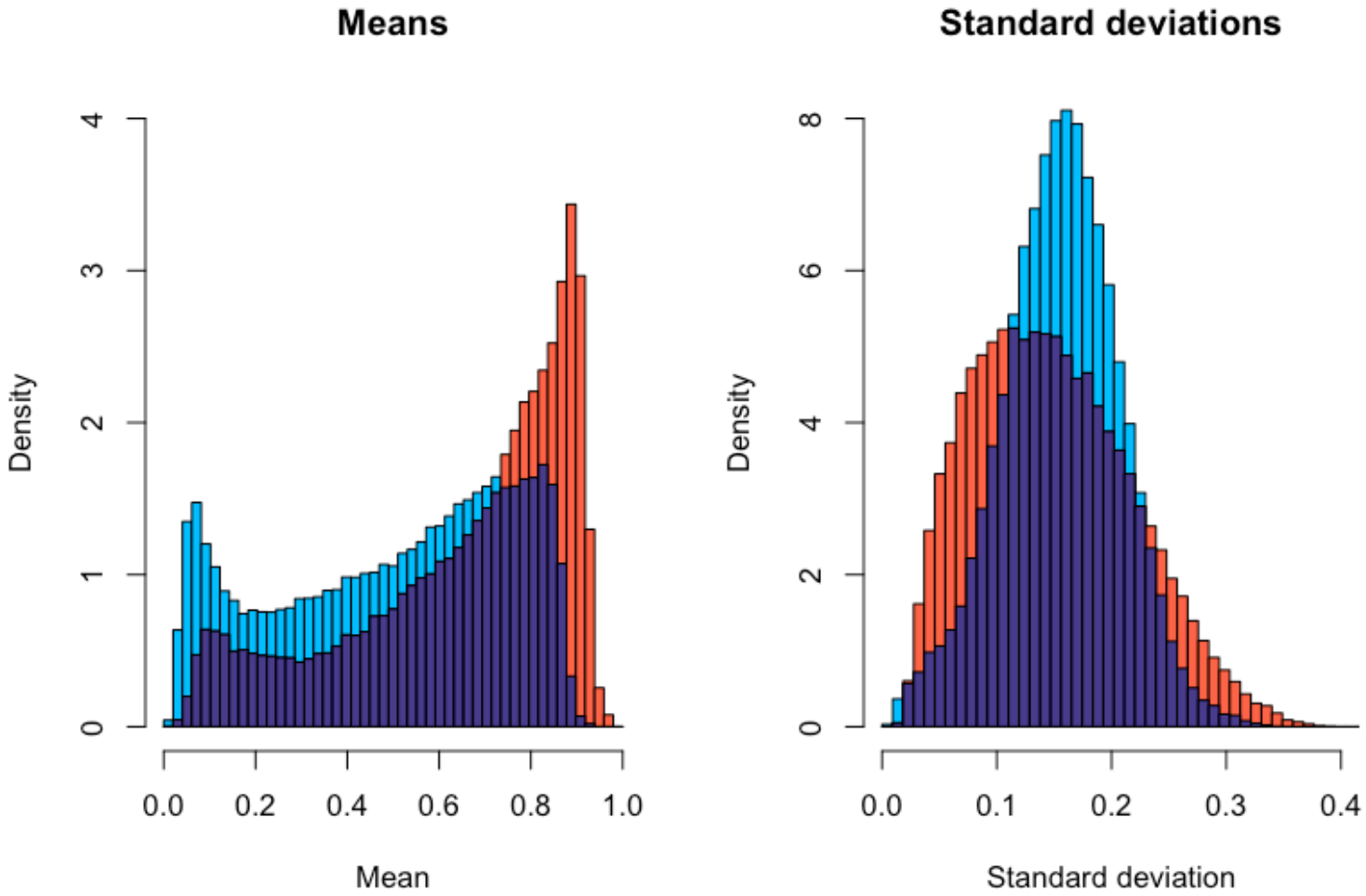




LGG vs. GBM

TCGA Glioma analysis

- CpGs with posterior probability of equality < 0.01

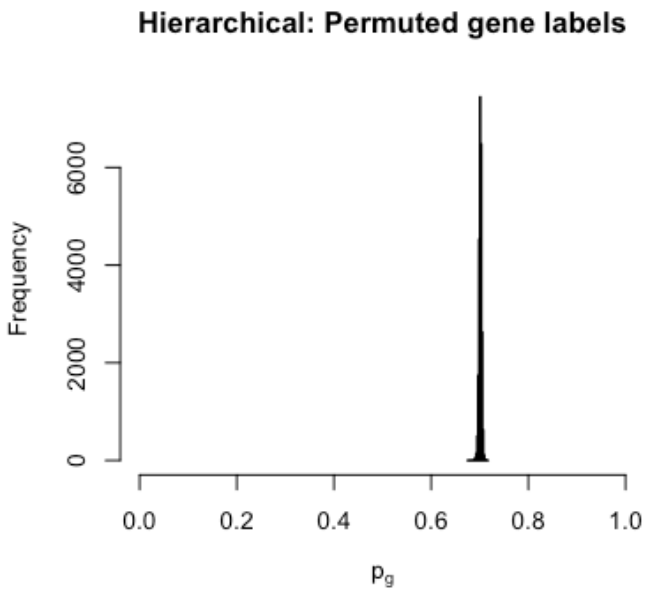


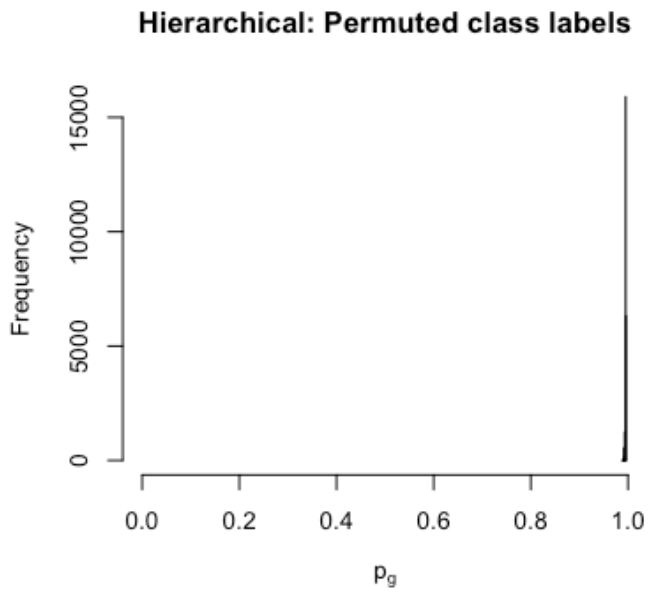
LGG vs. GBM

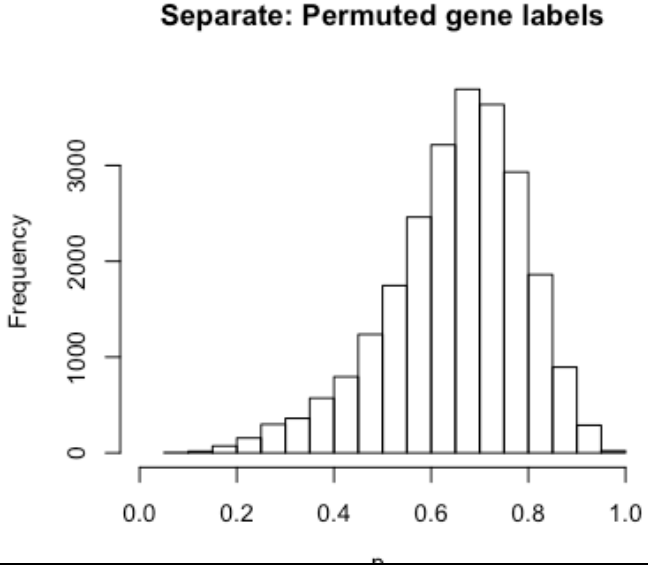
TCGA Glioma: Permutation

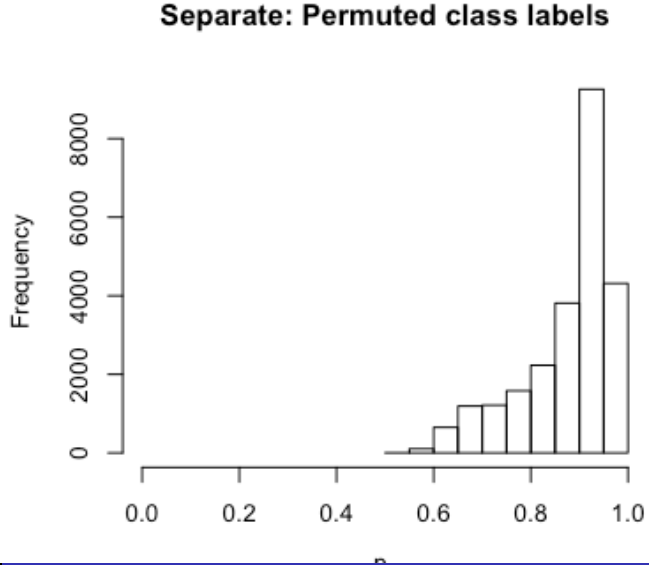
- Permute data under two different schemes:
  1. Randomly scramble the gene labels across CpGs
  2. Randomly scramble the class labels at each CpG
- Apply two methods to permutated datasets
  1. DP (hierarchical) prior for gene-level probabilities
  2. Independent (separate) inference of gene-level probabilities

TCGA Glioma analysis









Thank you!

- References:
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- R package `BayesianScreening`:
  - `github.com/lockEF/BayesianScreening`
- Email: elock@umn.edu